# A HEREDITARILY INDECOMPOSABLE ASYMPTOTIC $\ell_2$ BANACH SPACE

G. ANDROULAKIS, K. BEANLAND*

ABSTRACT. A Hereditarily Indecomposable asymptotic $\ell_2$ Banach space is constructed. The existence of such a space answers a question of B. Maurey and verifies a conjecture of W.T. Gowers.

## 1. Introduction

A famous open problem in functional analysis is whether there exists a Banach space $X$ such that every (bounded linear) operator on $X$ has the form $\lambda+K$ where $\lambda$ is a scalar and $K$ denotes a compact operator. This problem is usually called the "scalar-plus-compact" problem [11]. One of the reasons this problem has become so attractive is that by a result of N. Aronszajn and K.T. Smith [7], if a Banach space $X$ is a solution to the scalar-plus-compact problem then every operator on $X$ has a non-trivial invariant subspace and hence $X$ provides a solution to the famous invariant subspace problem. An important advancement in the construction of spaces with "few" operators was made by W.T. Gowers and B. Maurey [13],[14]. The ground breaking work [13] provides a construction of a space without any unconditional basic sequence thus solving, in the negative, the long standing unconditional basic sequence problem. The Banach space constructed in [13] is Hereditarily Indecomposable (HI), which means that no (closed) infinite dimensional subspace can be decomposed into a direct sum of two further infinite dimensional subspaces. It is proved in [13] that if $X$ is a complex HI space then every operator on $X$ can be written as $\lambda + S$ where $\lambda$ is a scalar and $S$ is strictly singular (i.e. the restriction of $S$ on any infinite dimensional subspace of $X$ is not an isomorphism). It is also shown in [13] that the same property remains true for the real HI space constructed in [13]. V. Ferenczi [15] proved that if $X$ is a complex HI space and $Y$ is an infinite dimensional subspace of $X$ then every operator from $Y$ to $X$ can be written as $\lambda i_Y + S$ where $i_Y : Y \to X$ is the inclusion map and $S$ is strictly singular. It was proved in [14] that, roughly speaking, given an algebra of operators satisfying certain conditions, there exists a Banach space $X$ such that for every infinite dimensional subspace $Y$, every operator from $Y$ to $X$ can be written as a strictly singular perturbation of a restriction to $Y$ of some element of the algebra.

The construction of the first HI space prompted researchers to construct HI spaces having additional nice properties. In other words people tried to "marry" the exotic structure of the HI spaces to the nice structure of classical Banach spaces. The reasons behind these efforts were twofold! Firstly, by producing more examples of HI spaces having additional well understood properties we can better understand how the HI property effects other behaviors of the space. Secondly, there is hope that endowing an HI space with additional

1991 *Mathematics Subject Classification.* Primary: 46B20, Secondary: 46B03.

*The present paper is part of the Ph.D thesis of the second author which is prepared at the University of South Carolina under the supervision of the first author.

nice properties could cause the strictly singular and compact operators on the space to coincide giving a solution to the scalar-plus-compact problem.

An open problem that has resisted the attempts of many experts is whether there exists a weak Hilbert HI Banach space. Recall that an infinite dimensional Banach space $X$ is a weak Hilbert Banach space [18],[19] if there exist positive numbers $\delta, C$ such that every finite dimensional space $E \subset X$ contains a subspace $F \subset E$ such that $\dim F \geq \delta \dim E$, the Banach-Mazur distance between $F$ and $\ell_2^{\dim F}$ is at most equal to $C$ and there is a projection $P : X \to F$ with $\|P\| \leq C$, ($\ell_2^n$ denotes the Hilbert space of dimension $n$). Operator theory on weak Hilbert spaces has been studied in [18],[19]. In particular, the Fredhold alternative has been established for weak Hilbert spaces.

Recall some standard notation: Given a Schauder basis $(e_n)$ of a Banach space, a sequence $(x_n)$ of non-zero vectors of $Span(e_m)_m$ is called a block basis of the $(e_i)$ if there exist successive subsets $F_1 < F_2 < \cdots$ of $\mathbb{N}$, (where for $E, F \subset \mathbb{N}$, $E < F$ means $\max E < \min F$), and a scalar sequence $(a_n)$ so that $x_n = \sum_{i \in F_n} a_i e_i$ for every $n \in \mathbb{N}$. We write $x_1 < x_2 < \cdots$ whenever $(x_n)$ is a block basis of $(e_i)$. If $x = \sum_n a_n e_n \in Span(e_m)_m$ then define the support of $x$ by $\operatorname{supp} x = \{i : a_i \neq 0\}$, and the range of $x$, $r(x)$, as the smallest interval of integers containing $\operatorname{supp} x$.

Some of the efforts that have been made in order to construct HI space possessing additional nice properties are the following. Gowers [12] constructed an HI space which has an asymptotically unconditional basis. A Schauder basis $(e_n)$ is called asymptotically unconditional if there exist a constant $C$ such that for any positive integer $m$, and blocks $(x_i)_{i=1}^m$ of $(e_n)$ with $m \leq x_1$ (i.e $m \leq \min \operatorname{supp} x_1$) and for any signs $(\varepsilon_i)_{i=1}^m \subset \{\pm 1\}$ we have

$$\Big\| \sum_{i=1}^m \varepsilon_i x_i \Big\| \leq C \Big\| \sum_{i=1}^m x_i \Big\|.$$

Maurey [17, page 141-142] asked whether there exists and asymptotic $\ell_p$ space for $1 < p < \infty$ and Gowers conjectured existenc the of such spaces in [12, page 112]. Recall that a Banach space $X$ having a Schauder basis $(e_n)$ is called asymptotic $\ell_2$ if there exists a constant $C$ such that for every $m \in \mathbb{N}$ and all blocks $(x_i)_{i=1}^m$ of $(e_n)_n$ with $m \leq x_1$ we have

$$\frac{1}{C}\Big( \sum_{i=1}^m \|x_i\|^2 \Big)^{\frac{1}{2}} \leq \Big\| \sum_{i=1}^m x_i \Big\| \leq C \Big( \sum_{i=1}^m \|x_i\|^2 \Big)^{\frac{1}{2}}.$$

In the present paper we construct an HI Banach space which is asymptotic $\ell_2$. Our approach closely uses the methods and techniques of the paper [9] of I. Gasparis. The norm of our space $X$ satisfies an upper $\ell_2$-estimate for blocks (i.e. there exists a constant $C$ such that for all blocks $(x_i)_{i=1}^m$ of $(e_n)_n$ we have $\| \sum_{i=1}^m x_i \| \leq C (\sum_{i=1}^m \|x_i\|^2)^{\frac{1}{2}}$). In particular our result strengthens a result of N. Dew [8] who constructed an HI space which satisfies an upper $\ell_2$-estimate (but not a lower estimate for blocks $(x_i)_{i=1}^m$ with $m \leq x_1$).

S.A. Argyros and I. Deliyanni [2] constructed an HI space which is asymptotic $\ell_1$. Recall that a Banach space $X$ having a Schauder basis $(e_n)$ is called asymptotic $\ell_1$ if there exists a positive constant $C$ such that for every $m \in \mathbb{N}$ and all blocks $(x_i)_{i=1}^m$ of $(e_n)_n$ with $m \leq x_1$ we have

$$\Big\| \sum_{i=1}^m x_i \Big\| \geq C \sum_{i=1}^m \|x_i\|.$$

Ferenczi [16] constructed a uniformly convex HI space. Argyros and V. Felouzis [3] showed that for every $p > 1$ there exists an HI space $X_p$ such that $\ell_p$ (or $c_0$ when $p = \infty$) is isomorphic to a quotient of $X_p$. In particular the dual space $X_p^*$ is not an HI space since it contains an isomorph of $\ell_q$ (for $1/p + 1/q = 1$). Argyros and A. Tolias [6] have constructed an HI space whose dual space is saturated with unconditional sequences.

Finally we mention that a Banach space $Y$ such that every operator on $Y$ can be written as $\lambda + S$ with $S$ strictly singular, has to be indecomposable (i.e. the whole space cannot be decomposable into the direct sum of two infinite dimensional subspaces) but not HI. Indeed, Argyros and A. Manoussakis [4],[5] have constructed such spaces $Y$ not containing an HI subspace.

## 2. The Construction of the Space $X$

In this section we construct a Banach space $X$. We will prove in section 3 that $X$ is asymptotic $\ell_2$ and in section 4 that $X$ is HI. The construction makes use of the Schreier families $S_\xi$ (for $\xi < \omega$) which are defined in the following way, [1]. Set $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$. After defining $S_\xi$ for $\xi < \omega$, set

$$S_{\xi+1} = \{\cup_{i=1}^n F_i : n \in \mathbb{N}, n \le F_1 < \cdots < F_n, F_i \in S_\xi\},$$

(here we assume that the empty set satisfies $\emptyset < F$ and $F < \emptyset$ for any set $F$). Important properties of the Schreier families is that they are hereditary (i.e if $F \in S_\xi$ and $G \subset F$ then $G \in S_\xi$), spreading (i.e if $(p_i)_{i=1}^n \in S_\xi$ and $p_i \le q_i$ for all $i \le n$ then $(q_i)_{i=1}^n \in S_\xi$), and they have the convolution property (i.e. if $F_1 < \cdots < F_n$ are each members of $S_\alpha$ such that $\{\min F_i : i \le n\}$ belongs to $S_\beta$ then $\cup_{i=1}^n F_i$ belongs to $S_{\alpha+\beta}$). For $E_i \subset \mathbb{N}$ we say $(E_i)_{i=1}^n$ is $S_\xi$-admissible if $E_1 < E_2 < \cdots < E_n$ and $(\min E_i)_i \in S_\xi$.

Let $[\mathbb{N}]$ denote the collection of infinite sequences of positive integers and for $M \in [\mathbb{N}]$ let $[M]$ denote the collection of infinite sequences of elements of $M$. Let $c_{00}$ denote the vector space of the finitely supported scalar sequences and $(e_n)$ denote the unit vector basis of $c_{00}$.

Using Schreier families we define repeated hierarchy averages similarly as in [9]. For $\xi < \omega$ and $M \in [\mathbb{N}]$, we define a sequence $([\xi]_n^M)_{n=1}^\infty$, of elements of $c_{00}$ whose supports are successive subsets of $M$, as follows:

For $\xi = 0$, let $[\xi]_n^M = e_{m_n}$ for all $n \in \mathbb{N}$, where $M = (m_n)$. Assume that $([\xi]_n^M)_{n=1}^\infty$ has been defined for all $M \in [\mathbb{N}]$. Set

$$[\xi + 1]_1^M = \frac{1}{\min M} \sum_{i=1}^{\min M} [\xi]_i^M.$$

Suppose that $[\xi + 1]_1^M < \cdots < [\xi + 1]_n^M$ have been defined. Let

$$M_n = \{m \in M : m > \max \operatorname{supp} [\xi + 1]_n^M\} \text{ and } k_n = \min M_n.$$

Set

$$[\xi + 1]_{n+1}^M = \frac{1}{k_n} \sum_{i=1}^{k_n} [\xi]_i^{M_n}.$$

For $x \in c_{00}$ let $(x(k))_{k \in \mathbb{N}}$ denote the coordinates of $x$ with respect to $(e_k)$ (i.e. $x = \sum_k x(k) e_k$). For $M \in [\mathbb{N}]$, $\xi < \omega$ and $n \in \mathbb{N}$ define $(\xi)_n^M \in c_{00}$ by $(\xi)_n^M(k) = \sqrt{[\xi]_n^M(k)}$ for all $k \in \mathbb{N}$. It is

proved in [10] that for every $M \in [\mathbb{N}]$ and $\xi < \omega$, $\sup\{\sum_{k\in F}[\xi]_1^M(k) : F \in S_{\xi-1}\} < \xi/\min M$. From this it follows that

$$\text{for every } \xi < \omega \text{ and } \varepsilon > 0 \text{ there exists } n \in \mathbb{N} \text{ such that for all } M \in [\mathbb{N}] \text{ with}$$
$$n \le \min M \text{ we have that } \sup\left\{\left(\textstyle\sum_{k\in F}((\xi)_1^M(k))^2\right)^{\frac{1}{2}} : F \in S_{\xi-1}\right\} < \varepsilon. \tag{1}$$

**Definition 2.1.** *Let $(u_n)_n$ be a normalized block basis of $(e_n)_n$, $\varepsilon > 0$ and $1 \le \xi < w$. Set $p_n = \min \operatorname{supp} u_n$ for all $n \in \mathbb{N}$ and $P = (p_n)$.*

1. *An $(\varepsilon, \xi)$ squared average of $(u_n)_n$ is any vector that can be written in the form $\sum_{n=1}^\infty (\xi)_1^R(p_n)u_n$, where $R \in [P]$ and $\sup\{(\sum_{k\in F}((\xi)_1^R(k))^2)^{\frac{1}{2}} : F \in S_{\xi-1}\} < \varepsilon$.*
2. *A normalized $(\varepsilon, \xi)$ squared average of $(u_n)_n$ is any vector $u$ of the form $u = v/\|v\|$ where $v$ is a $(\varepsilon, \xi)$ squared average of $(u_n)_n$. In the case where $\|v\| \ge 1/2$, $u$ is called a smoothly normalized $(\varepsilon, \xi)$ squared average of $(u_n)_n$.*

In order to define the asymptotic $\ell_2$ HI space $X$ we fix four sequences $M = (m_i)$, $L = (\ell_i)$, $F = (f_i)$ and $N = (n_i)$ of positive integers which are defined as follows: Let $M = (m_i)_{i\in\mathbb{N}} \in [\mathbb{N}]$ be such that $m_1 > 246$ and $m_i^2 < m_{i+1}$ for all $i \in \mathbb{N}$. Choose $L = (l_i)_{i\in\mathbb{N}} \in [\mathbb{N}]$ such that and $2^{l_i} > m_i$ for all $i \in \mathbb{N}$. Now choose and infinite sequences $N = (n_i)_{i\in\mathbb{N}\cup\{0\}}$ and $F = (f_i)_{i\in\mathbb{N}}$ such that $n_0 = 0$, $l_j(f_j + 1) < n_j$ for all $j \in \mathbb{N}$, $f_1 = 1$ and for $j \ge 2$,

$$f_j = \max\left\{\sum_{1\le i<j} \rho_i n_i : \rho_i \in \mathbb{N}\cup\{0\}, \prod_{1\le i<j} m_i^{\rho_i} < m_j^3\right\}. \tag{2}$$

We now define appropriate trees.

**Definition 2.2.** *A set $\mathcal{T}$ is called an appropriate tree if the following four conditions hold:*

1. *$\mathcal{T}$ is a finite set and each element of $\mathcal{T}$ (which is called a node of $\mathcal{T}$) is of the form $(t_1, \dots, t_{3n})$ where $n \in \mathbb{N}$, $t_{3i-2} \in M$ for $1 \le i < n$ (these nodes are called the $M$-entries of $(t_1, \dots, t_{3n})$), $t_{3n-2} = 0$, $t_{3i-1}$ is a finite subset of $\mathbb{N}$ for $1 \le i \le n$, and $t_{3i}$ is a rational number of absolute value at most equal to 1 for $1 \le i \le n$.*
2. *$\mathcal{T}$ is partially ordered with respect to the initial segment inclusion $\prec$, i.e. if $(t_1, \dots, t_{3n})$, $(s_1, \dots, s_{3m}) \in \mathcal{T}$ then $(t_1, \dots, t_{3n}) \prec (s_1, \dots, s_{3n})$ if $n < m$ and $t_i = s_i$ for $i = 1, \dots, 3n$. For $\alpha, \beta \in \mathcal{T}$ we also write $\alpha \preceq \beta$ to denote $\alpha \prec \beta$ or $\alpha = \beta$. For $\alpha \in \mathcal{T}$ the elements $\beta \in \mathcal{T}$ satisfying $\beta \prec \alpha$ (respectively $\alpha \prec \beta$) are called the predecessors (resp. successors) of $\alpha$. If $(t_1, \dots, t_{3n}) \in \mathcal{T}$ then the length of $(t_1, \dots, t_{3n})$ is denoted by $|(t_1, \dots, t_{3n})|$ and it is equal to $3n$. There exists a unique element of $\mathcal{T}$ which has length 3 and it is called the root of $\mathcal{T}$, and it is the minimum element of $\mathcal{T}$ with respect to $\prec$. Every element $\alpha \in \mathcal{T}$ except the root of $\mathcal{T}$ has a unique immediate predecessor which is denoted by $\alpha^-$. If $\alpha$ is the root of $\mathcal{T}$ set $\alpha^- = \emptyset$. If $(t_1, \dots, t_{3n}) \in \mathcal{T}$ then $(t_1, \dots, t_{3\ell}) \in \mathcal{T}$ for all $1 \le \ell \le n$. The nodes of $\mathcal{T}$ without successors are called terminal. If $\alpha \in \mathcal{T}$ is non-terminal, then the set of nodes $\beta \in \mathcal{T}$ with $\alpha \prec \beta$ and $|\beta| = |\alpha| + 3$ are called immediate successors of $\alpha$. Also $D_\alpha$ denotes the set of immediate successors of $\alpha$.*
3. *If $\alpha \in \mathcal{T}$ then the last three entrees of $\alpha$ will be denoted by $m_\alpha$, $I_\alpha$ and $\gamma_\alpha$ respectively. If $\alpha$ is the root of $\mathcal{T}$ then $m_\alpha$ (resp. $I_\alpha$) is called the weight (resp. the support of $\mathcal{T}$ denoted by $\operatorname{supp}(\mathcal{T})$). If $\alpha$ is a terminal node of $\mathcal{T}$ then $I_\alpha = \{p_\alpha\}$ for some $p_\alpha \in \mathbb{N}$.*

*If $\alpha$ is a non-terminal node of $\mathcal{T}$ then $I_\alpha = \cup\{I_\beta : \beta \in D_\alpha\}$ and for $\beta, \delta \in D_\alpha$ with $\beta \neq \delta$ we have either $I_\beta < I_\delta$ or $I_\delta < I_\beta$.*

(4) *If $\alpha \in \mathcal{T}$ is non-terminal and $m_\alpha = m_{2j}$ for some $j$, then $(I_\beta)_{\beta \in D_\alpha}$ is $S_{n_{2j}}$-admissible and $\sum_{\beta \in D_\alpha} \gamma_\beta^2 \leq 1$.*

Now set

$$G = \{\mathcal{T} : \mathcal{T} \text{ is an appropriate tree}\}.$$

We make the convention that the empty tree belongs to $G$.

If $\mathcal{T}_1, \mathcal{T}_2 \in G$ then we write $\mathcal{T}_1 < \mathcal{T}_2$ if $\operatorname{supp}(\mathcal{T}_1) < \operatorname{supp}(\mathcal{T}_2)$. If $\mathcal{T} \in G$ and $I$ is an interval of integers then we define the restriction of $\mathcal{T}$ on $I$, $\mathcal{T}|_I$, to denote the tree resulting from $\mathcal{T}$ by keeping only those $\alpha \in \mathcal{T}$ for which $I_\alpha \cap I \neq 0$ and replacing $I_\alpha$ by $I_\alpha \cap I$. It is easy to see that $\mathcal{T}|_I \in G$. It $\alpha \in \mathcal{T}$ set $\mathcal{T}_\alpha = \{\beta \setminus \alpha^- : \beta \in \mathcal{T}, \alpha \preceq \beta\}$ (for $\alpha = (t_1, \dots, t_{3n}) \prec \beta = (t_1, \dots, t_{3m})$ let $\beta \setminus \alpha = (t_{3n+1}, \dots, t_{3m})$). Clearly $\mathcal{T}_\alpha \in G$. For $\mathcal{T} \in G$ and $\alpha_0$ the root of $\mathcal{T}$, define $-\mathcal{T}$ by changing $\gamma_{\alpha_0}$ to $-\gamma_{\alpha_0}$ and keeping everything else in $\mathcal{T}$ unchanged.

Define an injection

$$\sigma : \{(\mathcal{T}_1 < \cdots < \mathcal{T}_n) : n \in \mathbb{N}, \mathcal{T}_i \in G \ (i \leq n)\} \to \{m_{2j} : j \in \mathbb{N}\}$$

such that $\sigma(\mathcal{T}_1, \cdots, \mathcal{T}_n) > w(\mathcal{T}_i)$ for all $1 < i \leq n$.

**Definition 2.3.** (1) *For $j \in \mathbb{N}$, a collection $(\mathcal{T}_\ell)_{\ell=1}^n \subset G$ is called $S_j$ admissible if $(\operatorname{supp} \mathcal{T}_\ell)_{\ell=1}^n$ is $S_j$-admissible.*

(2) *A collection of $S_{n_{2j+1}}$-admissible trees $(\mathcal{T}_\ell)_{\ell=1}^n \subset G$ is called $S_{n_{2j+1}}$-dependent if $w(\mathcal{T}_1) = m_{2j_1}$ for some $j_1 \geq j+1$ and $\sigma(\mathcal{T}_1, \cdots, \mathcal{T}_{i-1}) = w(\mathcal{T}_i)$ for all $2 \leq i \leq n$.*

(3) *Let $G_0 \subset G$. A collection of $S_{n_{2j+1}}$ admissible trees $(\mathcal{T}_\ell)_\ell^n \subset G$ is said to admit an $S_{n_{2j+1}}$-dependent extension in $G_0$ if there exist $k \in \mathbb{N} \cup \{0\}$ , $L \in \mathbb{N}$ and $\mathcal{R}_1 < \cdots < \mathcal{R}_{k+1} < \cdots < \mathcal{R}_{k+n} \in G_0$, where $\mathcal{R}_{k+i}|_{[L,\infty)} = \mathcal{T}_i$ for all $1 \leq i \leq n$.*

(4) *We say that $G_0 \subset G$ is self dependent if for all $j \in \mathbb{N}$, $\mathcal{T} \in G_0$ and $\alpha \in \mathcal{T}$ such that $m_\alpha = m_{2j+1}$, the family $\{\mathcal{T}_\beta : \beta \in D_\alpha\}$ admits an $S_{n_{2j+1}}$-dependent extension in $G_0$.*

A set $G_0 \subset G$ is symmetric if $-\mathcal{T} \in G_0$ whenever $\mathcal{T} \in G_0$; $G_0$ is closed under restriction to intervals if $\mathcal{T}|_J \in G_0$ whenever $\mathcal{T} \in G_0$ and $J \subset \mathbb{N}$ an interval.

**Definition 2.4.** *Let $\Gamma$ be the union of all non-empty, self-dependent, symmetric subsets of $G$ closed under restrictions to intervals such that for every $\mathcal{T} \in \Gamma$ and $\alpha \in \mathcal{T}$, if $m_\alpha = m_{2j+1}$ for some $j \in \mathbb{N}$ then:*

(1) *$\alpha$ is non-terminal.*

(2) *$D_\alpha = \{\beta_1, \cdots, \beta_n\}$ for some $n \in \mathbb{N}$ with $\mathcal{T}_{\beta_1} < \cdots < \mathcal{T}_{\beta_n}$ and there exist $k \in \mathbb{N} \cup \{0\}$, $L \in \mathbb{N}$ and an $S_{n_{2j+1}}$-dependent family $\mathcal{R}_1 < \cdots < \mathcal{R}_{k+1} < \cdots < \mathcal{R}_{k+n} \in \Gamma$, with $\mathcal{R}_{k+i}|_{[L,\infty)} = \mathcal{T}_i$ for $1 \leq i \leq n$.*

(3) *$(\gamma_i)_{i=1}^n$ is a non-increasing sequence of positive rationals such that $\sum_{i=1}^n \gamma_i^2 \leq 1$.*

**Notation 2.5.** *Let $\mathcal{T} \in \Gamma$ and $\alpha \in \mathcal{T}$.*

(1) *Define the height of the tree $\mathcal{T}$ by $o(\mathcal{T}) = \max\{|\beta| : \beta \in \mathcal{T}\}$.*

(2) *Let $m(\alpha) = \Pi_{\beta \prec \alpha} m_\beta$ if $|\alpha| > 3$, while $m(\alpha) = 1$ if $|\alpha| = 3$.*

(3) *If $m_\alpha = m_i$ for some $i \in \mathbb{N}$ set $n_\alpha = n_i$. Also set $n(\alpha) = \sum_{\beta \prec \alpha} n_\beta$ if $|\alpha| > 3$, while $n(\alpha) = 0$ if $|\alpha| = 3$.*

(4) *Let $\gamma(\alpha) = \Pi_{\beta \prec \alpha} \gamma_\beta$ if $|\alpha| > 3$, while $\gamma(\alpha) = \gamma_\alpha$ if $|\alpha| = 3$.*

Let $(e_n^*)_n$ denote the biorthogonal functionals to the unit vector basis of $c_{00}$. Given $\mathcal{T} \in \Gamma$, set

$$x_{\mathcal{T}}^* = \sum_{\alpha \in \max \mathcal{T}} \frac{\gamma(\alpha)\gamma_\alpha}{m(\alpha)} e_{p_\alpha}^*$$

where $\max \mathcal{T}$ is the set of terminal nodes of $\mathcal{T}$ and $I_\alpha = \{p_\alpha\}$ for $\alpha \in \max \mathcal{T}$.

Let $\mathcal{N} = \{x_{\mathcal{T}}^* : \mathcal{T} \in \Gamma\}$ and define $X$ to be completion of $c_{00}$ under the norm $\|x\| = \sup\{|x^*(x)| : x^* \in \mathcal{N}\}$.

Note that for each $\mathcal{T} \in \Gamma$ there is a unique norming functional $x_{\mathcal{T}}^* \in \mathcal{N} \subset \{x^* : \|x^*\| \leq 1\}$ thus set $w(x_{\mathcal{T}}^*) = w(\mathcal{T})$ and $\operatorname{supp}(x_{\mathcal{T}}^*) = \operatorname{supp}(\mathcal{T})$. We will often use the range of $x^* \in \mathcal{N}$, $r(x^*)$, which is the smallest interval containing $\operatorname{supp}(x^*)$. If $x^* \in \mathcal{N}$ and $I$ is an interval of integers define the restriction of $x^*$ on $I$, $x^*|_I$, by $x^*|_I(e_i) = x^*(e_i)$ if $i \in I$ and $x^*|_I(e_i) = 0$ if $i \notin I$. It is then obvious that if $\mathcal{T} \in \Gamma$ and $I$ is an interval of integers then $x_{\mathcal{T}}^*|_I = x_{\mathcal{T}|_I}^*$. For $j \in \mathbb{N}$ and $\mathcal{T}_1, \cdots, \mathcal{T}_n \in \Gamma$ we say that $(x_{\mathcal{T}_\ell}^*)_{\ell=1}^n$ is $S_{n_{2j+1}}$-dependent (or it admits an $S_{n_{2j+1}}$-dependent extension) if $(\mathcal{T}_\ell)_{\ell=1}^n$ is $S_{n_{2j+1}}$-dependent (or admits an $S_{n_{2j+1}}$-dependent extension). Also we say that a collection $(x_i)_{i=1}^n \subset c_{00}$ is $S_j$ admissible if $(\operatorname{supp} x_i)_{i=1}^n$ is $S_j$ admissible.

The maximality of $\Gamma$ implies the following:

**Remark 2.6.** (1) *$e_n^* \in \mathcal{N}$ for all $n \in \mathbb{N}$.*

(2) *For each $\mathcal{T} \in \Gamma$ and $\alpha \in \mathcal{T}$ the tree $\mathcal{T}_\alpha = \{\beta \setminus \alpha^- : \beta \in \mathcal{T}, \alpha \preceq \beta\}$ is in $\Gamma$.*

(3) *For every $x^* \in \mathcal{N}$ with $w(x^*) = m_{2j}$ for a some $j \in \mathbb{N}$ we can write*

$$x^* = \frac{1}{m_{2j}} \sum_\ell \gamma_\ell x_\ell^*$$

*for some $(\gamma_\ell)_\ell$ in $c_{00}$ with $\sum_\ell \gamma_\ell^2 \leq 1$ and $x_\ell^* \in \mathcal{N}$ where $(\operatorname{supp} x_\ell^*)_\ell$ is $S_{n_{2j}}$-admissible.*

(4) *For every $x^* \in \mathcal{N}$ with $w(x^*) = m_{2j+1}$ for a some $j \in \mathbb{N}$ we can write*

$$x^* = \frac{1}{m_{2j+1}} \sum_{\ell=1}^t \gamma_\ell x_\ell^*$$

*for some positive decreasing $(\gamma_\ell)_\ell$ in $c_{00}$ with $\sum_\ell \gamma_\ell^2 \leq 1$. Furthermore there exists $k \in \mathbb{N} \cup \{0\}, L \in \mathbb{N}$ and $y_1^* < \cdots < y_{k+1}^* < \cdots < y_{k+t}^*$ where $(y_\ell^*)_\ell$ is $S_{n_{2j+1}}$-dependent and $y_{k+i}^*|_{[L,\infty)} = x_i^*$ for all $i \leq t$.*

## 3. Preliminary Estimates

In this section we make some estimates similar to those in [9] that will be important in the proof that $X$ is H.I. First we show that $X$ is asymptotic $\ell_2$. Obviously $(e_n)_n$ is a bimonotone unit vector basis for $X$ since the linear span of $(e_n)_n$ is dense in $X$ and for finite intervals $I, J$ of integers with $I \subset J$ and scalars $(a_n)_n$ we have $\|\sum_{n \in I} a_n e_n\| \leq \|\sum_{n \in J} a_n e_n\|$ (this follows from the fact that $\Gamma$ is closed under restrictions to intervals).

We now introduce a short remark.

**Remark 3.1.** *If $x \in Span(e_n)_n$, $x^* = \frac{1}{m_i}\sum_j \gamma_j x_j^* \in \mathcal{N}$, $J = \{j : r(x_j^*) \cap r(x) \neq \emptyset\}$ then*

$$\frac{1}{m_i} \sum_j \gamma_j x_j^*(x) \leq \left(\sum_{j \in J} \gamma_j^2\right)^{\frac{1}{2}} \|x\|.$$

*Proof.* Indeed there exists an arbitrarily small $\eta > 0$ such that,

$$\Big(\sum_{m\in J}\gamma_m^2\Big)^{\frac{1}{2}} + \eta \in \mathbb{Q}.$$

For $j \in J$ let $\beta_j = \gamma_j/((\sum_{m\in J}\gamma_m^2)^{\frac{1}{2}} + \eta) \in \mathbb{Q}$. Notice that $(\sum_j \beta_j^2)^{\frac{1}{2}} \le 1$ and if $i$ is odd then we have that $(\beta_j)_j$ are non-decreasing and positive (since $(\gamma_j)_j$ are) and $(x_j^*)_{j\in J}$ has a dependent extension. Thus $1/m_i \sum_{j\in J}\beta_j x_j^* \in \mathcal{N}$, hence

$$\begin{aligned}\frac{1}{m_i}\sum_j \gamma_j x_j^*(x) &= \Bigg(\Big(\sum_{j\in J}\gamma_j^2\Big)^{\frac{1}{2}} + \eta\Bigg)\frac{1}{m_i}\sum_{j\in J}\beta_j x_j^*(x)\\ &\le \Bigg(\Big(\sum_{j\in J}\gamma_j^2\Big)^{\frac{1}{2}} + \eta\Bigg)\|x\|\end{aligned}$$

Since $\eta > 0$ is arbitrary, the result follows. □

The next proposition shows that the norm of $X$ satisfies an upper $\ell_2$-estimate for blocks.

**Proposition 3.2.** *If $(x_i)_{i=1}^m$ is a normalized block basis of $X$ then for any sequence of scalars $(a_i)_i$ the following holds:*

$$\Big\|\sum_{i=1}^m a_i x_i\Big\| \le \sqrt{3}\Big(\sum_{i=1}^m |a_i|^2\Big)^{\frac{1}{2}}.$$

*Proof.* For the purposes of this proposition define $\Gamma_n = \{\mathcal{T} \in \Gamma : o(\mathcal{T}) \le 3n\}$ and $\mathcal{N}_n = \{x_{\mathcal{T}}^* : \mathcal{T} \in \Gamma_n\}$. For $x \in c_{00}$ define $\|x\|_n = \sup\{x^*(x) : x^* \in \mathcal{N}_n\}$. Notice that the norm $\|\cdot\|$ of $X$ satisfies $\lim_{n\to\infty}\|x\|_n = \|x\|$. We will use induction on $n$ to verify that $\|\cdot\|_n$ satisfies the statement of the proposition.

For $n = 1$, $\mathcal{N}_n = \{\gamma e_m^* : \gamma \in \mathbb{Q}, |\gamma| \le 1,\ m \in \mathbb{N}\}$, and so the claim is trivial.

For the inductive step, let $x^* = 1/m_k \sum_j \gamma_j x_j^* \in \mathcal{N}_{n+1}$ (where $(x_j^*)_j \subset \mathcal{N}_n$ is $S_{n_k}$ admissible and $\sum \gamma_i^2 \le 1$) and let

$$Q(1) = \{1 \le i \le m :\ \text{there is exactly one } j \text{ such that } r(x_j^*) \cap r(x_i) \ne \emptyset\},$$

and $Q(2) = \{1, \dots, m\} \setminus Q(1)$. Now apply the functional $x^*$ to $\sum_{i=1}^m a_i x_i$ to obtain:

$$\begin{aligned}\frac{1}{m_k}\sum_j \gamma_j x_j^*\Big(\sum_{i=1}^m a_i x_i\Big) &= \frac{1}{m_k}\sum_j \gamma_j x_j^* \sum_{\substack{i\in Q(1)\\ r(x_j^*)\cap r(x_i)\ne\emptyset}} a_i x_i + \frac{1}{m_k}\sum_j \gamma_j x_j^* \sum_{i\in Q(2)} a_i x_i\\ &\le \frac{\sqrt{3}}{m_k}\sum_j \gamma_j\Big(\sum_{\substack{i\in Q(1)\\ r(x_j^*)\cap r(x_i)\ne\emptyset}} a_i^2\Big)^{\frac{1}{2}} + \sum_{i\in Q(2)} a_i \frac{1}{m_k}\sum_j \gamma_j x_j^*(x_i),\end{aligned} \tag{3}$$

by applying the induction hypothesis for $\sum_{\{i\in Q(1):r(x_j^*)\cap r(x_i)\}} a_i x_i$. The above estimate continues as follows,

$$
\begin{aligned}
&\leq \sum_j \gamma_j \Big( \sum_{\substack{i\in Q(1)\\ r(x_j^*)\cap r(x_i)\neq\emptyset}} a_i^2 \Big)^{\frac{1}{2}} + \sum_{i\in Q(2)} a_i \Big( \sum_{\{j:r(x_j^*)\cap r(x_i)\neq\emptyset\}} \gamma_j^2 \Big)^{\frac{1}{2}} \\
(4)\qquad &\leq \Big(\sum_j \gamma_j^2\Big)^{\frac{1}{2}} \Big( \sum_j \sum_{\substack{i\in Q(1)\\ r(x_j^*)\cap r(x_i)\neq\emptyset}} a_i^2 \Big)^{\frac{1}{2}} + \sum_{i\in Q(2)} a_i \Big( 2\sum_j \gamma_j^2 \Big) \leq \sqrt{3}\Big(\sum_i^m a_i^2\Big)^{\frac{1}{2}},
\end{aligned}
$$

where for the first inequality of (4) we used that $\sqrt{3} < m_1$ and Remark 3.1 and for the second inequality of (4) we used the Cauchy-Schwartz inequality and the fact that for each $j$ there are at most two values of $i \in Q(2)$ such that $r(x_j^*)\cap r(x_i) \neq \emptyset$. For the third inequality of (4) we used $\sum_\ell \gamma_\ell^2 \leq 1$. Combine (3) and (4) to finish the inductive step.

□

**Corollary 3.3.** *Let $(x_i)_{i=1}^n$ be a block basis of $X$ with $n \leq x_1$. Then for any sequence of scalars $(a_i)_i$ the following holds:*

$$\frac{1}{m_2}\Big(\sum_{i=1}^n |a_i|^2\Big)^{\frac{1}{2}} \leq \Big\|\sum_{i=1}^n a_i x_i\Big\| \leq \sqrt{3}\Big(\sum_{i=1}^n |a_i|^2)\Big)^{\frac{1}{2}}.$$

*Proof.* Let $(x_i)_{i=1}^n$ be a normalized block sequence of $(e_n)$ such that $n \leq x_1 < \cdots < x_n$ and scalars $(a_i)_{i=1}^n$. The upper inequality follows from Proposition 3.2. Note that, $(x_i)_{i=1}^n$ is $S_1$ admissible hence $S_{n_2}$ admissible. Find norm one functionals $(x_i^*)_{i=1}^n$ such that $r(x_i^*) \subset r(x_i)$ and $x_i^*(x_i) = 1$ for all $i \leq n$. To establish the lower inequality apply the functional

$$\frac{1}{m_2}\sum_{i=1}^n \Big(a_i \Big/ \Big(\sum_{i=1}^n a_i^2\Big)^{\frac{1}{2}}\Big) x_i^*$$

(whose norm is at most equal to one) to $\sum_{i=1}^n a_i x_i$.

□

The next lemma is a variation of the decomposition lemma found in [9] and will be used in the proof of Lemma 3.12 and Proposition 4.1

**Lemma 3.4.** *(Decomposition Lemma) Let $x^* \in \mathcal{N}$. Let $j \in \mathbb{N}$ be such that $w(x^*) < m_j$. Then there exists as $S_{f_j}$-admissible collection $(x_\alpha^*)_{\alpha\in L}$ and a sequence of scalars $(\lambda_\alpha)_{\alpha\in L}$ such that $L = \cup_{i=1}^3 L_i$ and:*

(1) $x^* = \sum_{\alpha\in L} \lambda_\alpha x_\alpha^*$.
(2) $\Big(\sum_{\alpha\in L_1} \lambda_\alpha^2\Big)^{\frac{1}{2}} \leq \frac{1}{m_j^2}$, $\Big(\sum_{\alpha\in L} \lambda_\alpha^2\Big)^{\frac{1}{2}} \leq \frac{1}{w(x^*)}$.
(3) $w(x_\alpha^*) \geq m_j$ *for* $\alpha \in L_2$.
(4) *For* $\alpha \in L_3$ *there exists* $|\gamma_\alpha| \leq 1$ *and* $p_\alpha \in \mathbb{N}$ *such that* $x_\alpha^* = \gamma_\alpha e_{p_\alpha}^*$.

*Proof.* Since $x^* \in \mathcal{N}$ there exists $\mathcal{T} \in \Gamma$ such that $x^* = x^*_{\mathcal{T}}$. Define three pairwise disjoint sets $L_1, L_2, L_3$ of nodes of $\mathcal{T}$ such that for every branch $\mathcal{B}$ of $\mathcal{T}$ (i.e. a maximal subset of $\mathcal{T}$ which is totally ordered with respect to $\prec$) there is a unique $\alpha \in \mathcal{B}$ with $\alpha \in \cup_{i=1}^3 L_i$. For every branch $\mathcal{B}$ of $\mathcal{T}$ choose a node $\alpha \in \mathcal{B}$ which is maximal with respect to $\prec$ such that $m(\alpha) < m_j^2$ and all $M$-entries of $\alpha^-$ are less than $m_j$. If $\alpha$ is non-terminal and $m_\alpha < m_j$ then $\alpha^+ \in L_1$, where $\alpha^+$ is the unique $\prec$-immediate successor of $\alpha$ in $\mathcal{B}$. Thus for $\alpha^+ \in L_1$ we have $m_j^2 \leq m(\alpha^+) < m_j^3$. If $\alpha$ is non-terminal and $m_\alpha \geq m_j$ then $\alpha \in L_2$. If $\alpha$ is terminal then $\alpha \in L_3$. Let $L = \cup_{i=1}^3 L_i$. For $\alpha \in L$ let

$$x^*_\alpha = x^*_{\mathcal{T}_\alpha} = \frac{m(\alpha)}{\gamma(\alpha)} x^*_{\mathcal{T}}|_{I_\alpha} \text{ and } \lambda_\alpha = \frac{\gamma(\alpha)}{m(\alpha)}.$$

Since $m_j^2 \leq m(\alpha)$ for $\alpha \in L_1$ we have

$$\left(\sum_{\alpha \in L_1} \lambda_\alpha^2\right)^{\frac{1}{2}} = \left(\sum_{\alpha \in L_1} \left(\frac{\gamma(\alpha)}{m(\alpha)}\right)^2\right)^{\frac{1}{2}} \leq \frac{1}{m_j^2}\left(\sum_{\alpha \in L_1} \gamma(\alpha)^2\right)^{\frac{1}{2}} \leq \frac{1}{m_j^2},$$

where the last inequality follows from Definition 2.2 (4) and Definition 2.4 (3). If $\alpha \in L_2$ then $w(\mathcal{T}_\alpha) \geq m_j$. If $\alpha \in L_3$ then $x^*_{\mathcal{T}_\alpha} = \gamma_\alpha e^*_{p_\alpha}$. Finally, since $m(\alpha) \geq w(x^*)$ for all $\alpha \in L$ we have,

$$\left(\sum_{\alpha \in L} \lambda_\alpha^2\right)^{\frac{1}{2}} = \left(\sum_{\alpha \in L} \left(\frac{\gamma(\alpha)}{m(\alpha)}\right)^2\right)^{\frac{1}{2}} \leq \frac{1}{w(x^*)}\left(\sum_{\alpha \in L} \gamma(\alpha)^2\right)^{\frac{1}{2}} \leq \frac{1}{w(x^*)}.$$

By applying the following Remark 3.5 (which also appears in [9]) to the set $\{I_\alpha : \alpha \in L\}$ we conclude that $(x^*_\alpha)_{\alpha \in L}$ is $S_p$-admissible where $p = \max\{n(\alpha) : \alpha \in L\} \leq f_j$ by (1). Thus $(x^*_\alpha)_{\alpha \in L}$ is $S_{f_j}$ admissible. □

**Remark 3.5.** *Let $\mathcal{T} \in \Gamma$. Let $F$ be a subset of $\mathcal{T}$ consisting of pairwise incomparable nodes. Then $\{I_\alpha : \alpha \in F\}$ is $S_p$-admissible, where $p = \max\{n(\alpha) : \alpha \in F\}$.*

*Proof.* Proceed by induction on $o(\mathcal{T})$. For $o(\mathcal{T}) = 3$ the assertion is trivial. Assume the claim for all $\mathcal{T} \in \Gamma$ such that $o(\mathcal{T}) < 3n$. Let $\mathcal{T}$ such that $o(\mathcal{T}) = 3n$ and $w(\mathcal{T}) = m_i$. If $|F| = 1$ the assertion is trivial, thus assume $|F| > 1$. Let $\alpha_0$ be the root of $\mathcal{T}$. Thus for all $\beta \in D_{\alpha_0}$ the claim holds for $\mathcal{T}_\beta$. For each $\beta \in D_{\alpha_0}$ define, $F_\beta = \{\alpha \setminus \alpha_0 : \alpha \in F, \beta \preceq \alpha\} \subset \mathcal{T}_\beta$. We know that for every $\beta \in D_{\alpha_0}$ we have that $\{I_\alpha : \alpha \in F_\beta\}$ is $S_{p_\beta}$ admissible where $p_\beta = \max\{n_\beta(\alpha) : \alpha \in F_\beta\}$ and for every $\alpha \in \mathcal{T}_\beta$,

$$n_\beta(\alpha) = \sum_{\substack{\gamma \in \mathcal{T}_\beta \\ \gamma \prec \alpha \setminus \alpha_0}} n_\gamma = \sum_{\gamma \in \mathcal{T}, \gamma \prec \alpha} n_\alpha - n_i = n(\alpha) - n_i.$$

Thus $\{I_\alpha : \alpha \in F_\beta\}$ is $S_{n(\alpha)-n_i}$ admissible for all $\beta \in D_{\alpha_0}$. Also $\{I_\beta : \beta \in D_{\alpha_0}\}$ is $S_{n_i}$ admissible so we use the convolution property of Schreier families to conclude that $\{I_\alpha : \alpha \in F\}$ is $S_p$ admissible. □

**Lemma 3.6.** *Let $(u_n)$ be a normalized block basis of $(e_n)$. Let $j \in 2\mathbb{N}$ and let $y$ be an $(\varepsilon, f_j + 1)$ squared average of $(u_n)$ with $\varepsilon < 1/m_j$. Let $(x^*_\ell)_\ell \in \mathcal{N}$ be $S_\xi$-admissible, $\xi \leq f_j$ and $(\gamma_\ell)_\ell$ in $c_{00}$. Then,*

$$\sum_{\ell} \gamma_\ell x_\ell^*(y) \leq 5\Big(\sum_{\{\ell : r(x_\ell^*)\cap r(y)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac{1}{2}}.$$

*Proof.* Let $p_n = \min\operatorname{supp} u_n$, $R \in [(p_n)]$ and $y = \sum_n (f_j+1)_1^R(p_n)u_n$. Define

$$Q(1) = \{i : \text{ there is exactly one } \ell \text{ such that } r(x_\ell^*)\cap r(u_n)\neq\emptyset\} \text{ and}$$
$$Q(2) = \{i : \text{ there are at least two } \ell\text{'s such that } r(x_\ell^*)\cap r(u_n)\neq\emptyset\}.$$

$$\begin{aligned}
\Big(\sum_{\ell}\gamma_\ell x_\ell^*\Big)\Big(\sum_{n\in Q(1)}(f_j+1)_1^R(p_n)u_n\Big) &\leq \sum_{\ell}|\gamma_\ell|\Big|x_\ell^*\Big(\sum_{\substack{n\in Q(1)\\ r(x_\ell^*)\cap r(u_n)\neq\emptyset}}(f_j+1)_1^R(p_n)u_n\Big)\Big|\\
&\leq \sum_{\{\ell : r(x_\ell^*)\cap r(y)\neq\emptyset\}}|\gamma_\ell|\sqrt{3}\Big(\sum_{\substack{n\in Q(1)\\ r(x_\ell^*)\cap r(u_n)\neq\emptyset}}((f_j+1)_1^R(p_n))^2\Big)^{\frac{1}{2}} \qquad (5)\\
&\leq \sqrt{3}\Big(\sum_{\{\ell : r(x_\ell^*)\cap r(y)\neq\emptyset\}}\gamma_\ell^2\Big)^{\frac{1}{2}},
\end{aligned}$$

where for the second inequality we used Proposition 3.2 and for the third inequality we used the Cauchy-Schwartz inequality. For $n$'s in $Q(2)$,

$$\begin{aligned}
\Big|\sum_{\ell}\gamma_\ell x_\ell^*\Big(\sum_{n\in Q(2)}(f_j+1)_1^R(p_n)u_n\Big| &\leq \sum_{n\in Q(2)}(f_j+1)_1^R(p_n)\Big)\Big|\frac{1}{m_j}\sum_{\{\ell : r(x_\ell^*)\cap r(u_n)\neq\emptyset\}}\gamma_\ell x_\ell^*(u_n)\Big|m_j\\
&\leq \sum_{n\in Q(2)}(f_j+1)_1^R(p_n)\Big(\sum_{\{\ell : r(x_\ell^*)\cap r(u_n)\neq\emptyset\}}\gamma_\ell^2\Big)^{\frac{1}{2}}m_j \qquad (6)\\
&\leq \Big(\sum_{n\in Q(2)}((f_j+1)_1^R(p_n))^2\Big)^{\frac{1}{2}}\Big(\sum_{n\in Q(2)}\sum_{\{\ell : r(x_\ell^*)\cap r(u_n)\neq\emptyset\}}\gamma_\ell^2\Big)^{\frac{1}{2}}m_j\\
&\leq 2\Big(2\sum_{\{\ell : r(x_\ell^*)\cap r(y)\neq\emptyset\}}\gamma_\ell^2\Big)^{\frac{1}{2}},
\end{aligned}$$

where for the second inequality we used Remark 3.1 and that $j$ is even. For the third inequality we used the fact that $(p_n)_{n\in Q(2)} \in 2S_\xi$ (i.e. the union of two sets each which belongs to $S_\xi$), $\xi < f_j$, $\varepsilon < 1/m_j$ and the fact that for every $\ell$ there are at most two values of $n \in Q(2)$ such that $r(x_\ell^*)\cap r(u_n)\neq\emptyset$. Combining (5) and (6) we obtain the desired result since $2\sqrt{2}+\sqrt{3} < 5$.

$\square$

**Lemma 3.7.** *Let $(u_n)$ be a normalized block basis of $(e_n)$. Let $\varepsilon > 0$ and $j$ be an even integer. Then there a exists smoothly normalized $(\varepsilon, f_j+1)$ squared average of $(u_n)$.*

*Proof.* Let $P = (p_n)$ such that $p_n = \min\operatorname{supp} u_n$ for all $n \in \mathbb{N}$. By (1) assume without loss of generality that for all $R \in [P]$, $\sup\{\sum_{k\in F}((f_j+1)_1^R(k))^2)^{\frac{1}{2}} : F \in S_{f_j}\} < \varepsilon$. Suppose that the claim is false. For $1 \le r \le l_j$ construct normalized block bases $(u_i^r)_i$ of $(u_n)$ as follows: Set

$$u_i^1 = \sum_n (f_j+1)_i^P(p_n)u_n.$$

It must be the case that $\|u_i^1\| < 1/2$ for all $i \in \mathbb{N}$. Now for each $1 < r \le \ell_j$, if $(u_n^{r-1})_n$ has been defined let $p_i^{r-1} = \min\operatorname{supp} u_i^{r-1}$, $P_{r-1} = (p_i^{r-1})$ and

$$u_i^r = \sum_n (f_j+1)_i^{P_{r-1}}(p_n^{r-1})\frac{u_n^{r-1}}{\|u_n^{r-1}\|}.$$

For all $r$ and $i$, $\|u_i^r\| < 1/2$. Write $u_1^{\ell_j} = \sum_{n\in F} a_n u_n$ for some finite set $F \subset \mathbb{N}$ and $a_n > 0$ with $(u_n)_{n\in F}$ being $S_{(f_j+1)\ell_j}$ -admissible and $(\sum_{n\in F} a_n^2)^{\frac{1}{2}} \ge 2^{\ell_j - 1}$. For $n \in F$ let $u_n^* \in X^*$, $\|u_n^*\| = u_n^*(u_n) = 1$ and $\operatorname{supp} u_n^* \subset r(u_n)$. Set

$$x^* = \frac{1}{m_j}\sum_{n\in F}\left(a_n \Big/ \left(\sum_{m\in F} a_m^2\right)^{\frac{1}{2}}\right)u_n^*.$$

Since $(f_j+1)\ell_j < n_j$ and $j$ is even, we have that $\|x^*\| \le 1$. Thus

$$\frac{1}{2} > \|u_1^{\ell_j}\| \ge x^*(u_1^{\ell_j}) = \frac{1}{m_j}\sum_{n\in F}\frac{a_n}{\left(\sum_{m\in F} a_m^2\right)^{\frac{1}{2}}}u_n^*\left(\sum_{n\in F} a_n u_n\right) \ge \frac{2^{\ell_j-1}}{m_j},$$

contradicting that $m_j \le 2^{\ell_j}$.

□

**Lemma 3.8.** *Let $(u_n)_n$ be a normalized block basis of $(e_n)_n$ and $j_0 \in \mathbb{N}$. Suppose that $(y_k)_k$ is a block basis of $(u_n)_n$ so that $y_k$ is a smoothly normalized $(\varepsilon_k, f_{2j_k}+1)$ squared average of $(u_n)_n$ with $\varepsilon_k < 1/m_{2j_k}$ and $j_0 < 2j_1 < 2j_2 < \cdots$. Let $(x_m^*)_m \subset \mathcal{N}$ be $S_\xi$ admissible, $\xi < n_{j_0}$ and $(\gamma_m)_m, (\beta_k)_k \in c_{00}$. Then*

$$\sum_m \gamma_m x_m^*\left(\sum_k \beta_k y_k\right) \le 22\left(\sum_m \gamma_m^2\right)^{\frac{1}{2}}\left(\sum_k \beta_k^2\right)^{\frac{1}{2}}.$$

*Proof.* Define the following two sets,

$$Q(1) = \{k : \text{ there is exactly one } m \text{ such that } r(x_m^*) \cap r(y_k) \neq \emptyset\},$$

$$Q(2) = \{k : \text{ there are at least two } m\text{'s such that } r(x_m^*) \cap r(y_k) \neq \emptyset\}.$$

$$\Big|\sum_m \gamma_m x_m^*\Big(\sum_{\{k\in Q(1):r(x_m^*)\cap r(y_k)\neq\emptyset\}}\beta_k y_k\Big)\Big| + \Big|\sum_m \gamma_m x_m^*\Big(\sum_{k\in Q(2)}\beta_k y_k\Big)\Big|$$
$$\leq \sum_m |\gamma_m|\sqrt{3}\Big(\sum_{\{k\in Q(1):r(x_m^*)\cap r(y_k)\neq\emptyset\}}\beta_k^2\Big)^{\frac12} + \sum_{k\in Q(2)}|\beta_k|\Big|\sum_{\{m:r(x_m^*)\cap r(y_k)\neq\emptyset\}}\gamma_m x_m^*(y_k)\Big|$$
$$\leq 2\Big(\sum_m \gamma_m^2\Big)^{\frac12}\Big(\sum_m\sum_{\{k\in Q(1):r(x_m^*)\cap r(y_k)\neq\emptyset\}}\beta_k^2\Big)^{\frac12} + 10\sum_{k\in Q(2)}|\beta_k|\Big(\sum_{\{m:r(x_m^*)\cap r(y_k)\neq\emptyset\}}\gamma_m^2\Big)^{\frac12}$$
$$\leq 2\Big(\sum_m \gamma_m^2\Big)^{\frac12}\Big(\sum_k \beta_k^2\Big)^{\frac12} + 20\Big(\sum_m \gamma_m^2\Big)^{\frac12}\Big(\sum_k \beta_k^2\Big)^{\frac12} \leq 22\Big(\sum_k \beta_k^2\Big)^{\frac12}.$$

For the first inequality we used Proposition 3.2. For the second inequality we applied the Cauchy-Schwartz inequality in the first term of the sum and used the fact that $\xi < n_{j_0} < f_{2j_k}$ for all $k$ to apply Lemma 3.6 in the second term of the sum. The "10" in the second part of the second inequality comes from the fact that $y_k$ is smoothly normalized. For the third inequality we used the Cauchy-Schwartz inequality. The "20" after the third inequality comes from the fact that for every $m$ there are at most two values of $k \in Q(2)$ such that $r(x_m^*)\cap r(y_k)\neq\emptyset$. $\square$

**Lemma 3.9.** *Let $(u_n)_n$ be a normalized block basis of $(e_n)_n$. Suppose that $(y_j)_j$ is a block basis of $(u_n)_n$ so that $y_j$ is a smoothly normalized $(\varepsilon_j, f_{2j}+1)$ squared average of $(u_n)_n$ with $\varepsilon_j < 1/m_{2j}$. Then there exists a subsequence $(y_j)_{j\in I}$ of $(y_j)_j$ such that for every $j_0\in\mathbb{N}$, $j_1, j_2, \ldots \in I$ with $j_0 < 2j_1 < 2j_2 < \ldots$, $x^*\in\mathcal{N}$ with $w(x^*)\geq m_{j_0}$ and scalars $(\beta_j)_j\in c_{00}$ we have that:*

(1) *If $w(x^*) < m_{j_1}$ then*
$$x^*\Big(\sum_k \beta_k y_{j_k}\Big) < \frac{5}{m_e}\Big(\sum_{\{k:r(x^*)\cap r(y_{j_k})\neq\emptyset\}}\beta_k^2\Big)^{\frac12},$$
*where $m_e = m_{j_0}$ if $w(x^*) = m_{j_0}$ and $m_e = m_{j_0}^2$ if $w(x^*) > m_{j_0}$.*

(2) *If $m_{2j_s}\leq w(x^*) < m_{2j_{s+1}}$ for some $s\geq 1$ then*
$$x^*\Big(\sum_{k\neq s}\beta_k y_{j_k}\Big) < \frac{5}{m_{j_0}^2}\Big(\sum_{\{k\neq s:r(x^*)\cap r(y_{j_k})\neq\emptyset\}}\beta_k^2\Big)^{\frac12}.$$

*Proof.* Lemma 3.7 assures the existence the block sequence $(y_j)_j$ such that each $y_j$ is a smoothly normalized $(\varepsilon_j, f_{2j}+1)$ squared average of $(u_n)_n$. Let $T = (t_n)$, where $t_n = \min\operatorname{supp} u_n$. Choose $I = (j'_k)_k \in [\mathbb{N}]$ such that, $j'_1$ is an arbitrary integer, $\sum_{i>k}\varepsilon_{j'_i}^2 < \varepsilon_{j'_k}^2$ and

$$\Big(\sum_{i<k}\|y_{j'_i}\|_{\ell_1}^2\Big)^{\frac12} < \frac{m_{2j'_k}}{m_{2j'_{k-1}}}. \tag{7}$$

Let $j_0\in\mathbb{N}$, $j_1, j_2, \ldots \in I$ with $j_0 < 2j_1 < 2j_2 < \ldots, (\beta_k)\in c_{00}$ and $x^*\in\mathcal{N}$ such that $m_{j_0}\leq w(x^*) < m_{2j_1}$. By definition $y_{j_k} = v_{j_k}/\|v_{j_k}\|$, where $v_{j_k} = \sum_n (f_{2j_k}+1)_1^{R_k}(t_n)u_n$ and

$R_k \in [(t_n)]$ is chosen as in Definition 2.1. Let $x^* = 1/m_i \sum_\ell \gamma_\ell x^*_\ell$ for some $i$ where $\sum_\ell \gamma_\ell^2 \leq 1$, $(x^*_\ell)_\ell$ is $S_{n_i}$ admissible and $i < 2j_1$. Define the following two sets.

$$Q(1) = \{n : \text{ there is exactly one } \ell \text{ such that } r(x^*_\ell) \cap r(u_n) \neq \emptyset\},$$

$$Q(2) = \{n : \text{ there are at least two } \ell\text{'s such that } r(x^*_\ell) \cap r(u_n) \neq \emptyset\}.$$

We proceed with the case $n \in Q(1)$.

(8)

$$\begin{aligned}
&\frac{1}{w(x^*)} \sum_\ell \gamma_\ell x^*_\ell \Bigg( \sum_k \frac{\beta_k}{\|v_{j_k}\|} \sum_{\{n \in Q(1) : r(x^*_\ell) \cap r(u_n) \neq \emptyset\}} (f_{2j_k} + 1)_1^{R_k}(t_n) u_n \Bigg) \\
&\quad \leq \frac{2}{w(x^*)} \sum_\ell |\gamma_\ell| \Bigg| x^*_\ell \Bigg( \sum_{\{k : r(x^*_\ell) \cap r(y_{j_k}) \neq \emptyset\}} \beta_k \sum_{\{n \in Q(1) : r(x^*_\ell) \cap r(u_n) \neq \emptyset\}} (f_{2j_k} + 1)_1^{R_k}(t_n) u_n \Bigg) \Bigg| \\
&\quad \leq \frac{2\sqrt{3}}{w(x^*)} \sum_\ell |\gamma_\ell| \Bigg( \sum_{\{k : r(x^*_\ell) \cap r(y_{j_k}) \neq \emptyset\}} \sum_{\{n \in Q(1) : r(x^*_\ell) \cap r(u_n) \neq \emptyset\}} \beta_k^2 ((f_{2j_k} + 1)_1^{R_k}(t_n))^2 \Bigg)^{\frac{1}{2}} \\
&\quad \leq \frac{4}{w(x^*)} \Big( \sum_\ell \gamma_\ell^2 \Big)^{\frac{1}{2}} \Bigg( \sum_{\{k : r(x^*_\ell) \cap r(v_{j_k}) \neq \emptyset\}} \beta_k^2 \Bigg)^{\frac{1}{2}} \leq \frac{3}{w(x^*)} \Bigg( \sum_{\{k : r(x^*) \cap r(y_{j_k}) \neq \emptyset\}} \beta_k^2 \Bigg)^{\frac{1}{2}}
\end{aligned}$$

where for the first and second inequalities we used Proposition 3.2 and the fact that $\|v_{j_k}\| < 1/2$. For the third inequality we used that $2\sqrt{3} \leq 4$ and the Cauchy-Schwartz inequality. Notice that since $w(x^*) > m_{j_0}$ then we have $4/w(x^*) < 1/m_{j_0}^2$. For $n \in Q(2)$ we have,

(9)

$$\begin{aligned}
&\frac{1}{m_i} \sum_\ell \gamma_\ell x^*_\ell \sum_k \beta_k \frac{\sum_{n \in Q(2)} (f_{2j_k} + 1)_1^{R_k}(t_n) u_n}{\|v_{j_k}\|} \\
&\quad \leq 2 \sum_{\{k : r(x^*) \cap r(y_{j_k}) \neq \emptyset\}} |\beta_k| \sum_{n \in Q(2)} (f_{2j_k} + 1)_1^{R_k}(t_n) \Bigg| \frac{1}{m_i} \sum_\ell \gamma_\ell x^*_\ell(u_n) \Bigg| \\
&\quad \leq 2 \sum_{\{k : r(x^*) \cap r(y_{j_k}) \neq \emptyset\}} |\beta_k| \sum_{n \in Q(2)} (f_{2j_k} + 1)_1^{R_k}(t_n) \Bigg( \sum_{\{\ell :\ r(x^*_\ell) \cap r(u_n) \neq \emptyset\}} \gamma_\ell^2 \Bigg)^{\frac{1}{2}} \\
&\quad \leq 2 \sum_{\{k : r(x^*) \cap r(y_{j_k}) \neq \emptyset\}} |\beta_k| \Bigg( \sum_{n \in Q(2)} ((f_{2j_k} + 1)_1^{R_k}(t_n))^2 \Bigg)^{\frac{1}{2}} \Bigg( \sum_{n \in Q(2)} \sum_{\{\ell :\ r(x^*_\ell) \cap r(u_n) \neq \emptyset\}} \gamma_\ell^2 \Bigg)^{\frac{1}{2}},
\end{aligned}$$

where for the first inequality we used that $\|v_{j_k}\| > 1/2$, for the second inequality we used Remark 3.1 and for the third inequality we used the Cauchy-Schwartz inequality. Note that $(u_n)_{n \in Q(2)}$ is $2S_{n_i}$ admissible (i.e. it can be written as a union of two sets each of which is

$S_{n_i}$ admissible) and $n_i \leq f_{2j_1}$. Also note that for every $\ell$ there are at most two values of $n \in Q(2)$ such that $r(x_\ell^*) \cap r(u_n) \neq \emptyset$, to continue (9) as follows:

$$\begin{aligned}
&\leq 4 \sum_{\{k: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} \beta_k \varepsilon_{j_k} \Big(\sum_\ell \gamma_\ell^2\Big)^{\frac12} \leq 4 \Big(\sum_{\{k: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} \beta_k^2\Big)^{\frac12} \Big(\sum_{k\geq 1} \varepsilon_{j_k}^2\Big)^{\frac12} \\
&\leq 8\varepsilon_{j_1} \Big(\sum_{\{k: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} \beta_k^2\Big)^{\frac12} \leq \frac{4}{m_{j_1}} \Big(\sum_{\{k: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} \beta_k^2\Big)^{\frac12}.
\end{aligned} \tag{10}$$

Obviously (8),(9) and (10) finish the proof of part (1). Assume for $s \geq 1$, $m_{2j_s} \leq w(x^*) < m_{2j_{s+1}}$. Estimate $x^*(\sum_{k>s} \beta_k y_{j_k})$ similarly to (8),(9) and (10) where we replace $m_{2j_1}$ by $m_{2j_{s+1}}$. Estimate $x^*(\sum_{k<s} \beta_k y_{j_k})$ as follows:

$$\begin{aligned}
\frac{1}{w(x^*)} \sum_\ell \gamma_\ell x_\ell^* \sum_{k<s} \beta_k y_{j_k} &= \sum_{\{k<s: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} \Big|\beta_k \frac{1}{w(x^*)} \sum_\ell \gamma_\ell x_\ell^*(y_{j_k})\Big| \\
&\leq \sum_{\{k<s: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} |\beta_k| \frac{1}{w(x^*)} \|y_{j_k}\|_{\ell_1} \text{ (since } \|\sum_\ell \gamma_\ell x_\ell^*\|_\infty \leq 1) \\
&\leq \Big(\sum_{\{k<s: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} \beta_k^2\Big)^{\frac12} \frac{1}{w(x^*)} \Big(\sum_{\{k<s: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} \|y_{j_k}\|_{\ell_1}^2\Big)^{\frac12} \\
&\leq \Big(\sum_{\{k<s: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} \beta_k^2\Big)^{\frac12} \frac{1}{w(x^*)} \frac{m_{j_s}}{m_{j_{s-1}}} \leq \frac{1}{m_{j_{s-1}}} \Big(\sum_{\{k<s: r(x^*)\cap r(y_{j_k})\neq\emptyset\}} \beta_k^2\Big)^{\frac12},
\end{aligned}$$

where the third inequality comes from equation (7). This finishes the proof of part (2). □

**Remark 3.10.** *Lemma 3.9 will be used several times as follows: Given a normalized block sequence $(u_n)$ of $(e_n)$, Lemma 3.7 will guarantee the existence of a block sequence $(y_j)$ of $(e_n)$ such that $y_j$ is a smoothly normalized $(\varepsilon_j, f_{2j}+1)$ squared average of $(u_n)$ and $\varepsilon_j < 1/m_{2j}$ for all $j \in \mathbb{N}$. Choose a subsequence $(y_j)_{j\in I}$ of $(y_j)_j$ to satisfy the conclusion of Lemma 3.9. Let $j_0 \in \mathbb{N}$ and $j_1, j_2, \ldots \in I$ with $j_0 < 2j_1 < 2j_2 < \ldots$. Let $p_k = \min \mathrm{supp}\,(y_{j_k})$ for all $k \in \mathbb{N}$, $R \in [(p_k)]$,*

$$y = \sum_k (n_{j_0})_1^R(p_k) y_{j_k} \text{ and } g = \frac{y}{\|y\|}$$

*be a normalized $(1/m_{j_0}^2, n_{j_0})$ squared average of $(y_j)_{j\in I}$. Then the conclusion of Lemma 3.9 will be valid for "$\beta_k$"$= (n_{j_0})_1^R(p_k)$ and for all $x^* \in \mathcal{N}$ with $w(x^*) \geq m_{j_0}$.*

**Lemma 3.11.** *Let $(y_j)_{j\in I}$, $j \in 2\mathbb{N}$ and a normalized $(1/m_{j_0}^2)$ squared average $g$ of $(y_j)_{j\in I}$ be chosen as in Remark 3.10. Then for any $S_\xi$ admissible family $(x_\ell^*)_\ell \subset \mathcal{N}$, $\xi < n_{j_0}$ where $w(x_\ell^*) \geq m_{j_0}$ for all $\ell$ and $(\gamma_\ell)_\ell \in c_{00}$, we have*

$$\sum_\ell \gamma_\ell x_\ell^*(g) < \frac{47}{m_{j_0}} \Big(\sum_{\{\ell:\ w(x_\ell^*)>m_{j_0},\ r(x_\ell^*)\cap r(u)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12} + 6\Big(\sum_{\{\ell:\ w(x_\ell^*)=m_{j_0},\ r(x_\ell^*)\cap r(u)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12}. \tag{11}$$

*Proof.* Let $g = y/\|y\|$ and note that since $j_0$ is even, $\|y\| \geq 1/m_{j_0}$ where $y = \sum_k (n_{j_0})_1^R(p_k) y_{j_k}$ for $p_k = \min\operatorname{supp} y_{j_k}$ for all $k \in \mathbb{N}$ and $R \in [(p_k)]$. Thus

$$\sum_\ell \gamma_\ell x_\ell^*(g) \leq m_{j_0} \sum_\ell |\gamma_\ell||x_\ell^*(y)| \tag{12}$$

Let $B = \{\ell : w(x_\ell^*) > m_{j_0}\}$ and $E = \{\ell : w(x_\ell^*) = m_{j_0}\}$. Define,

$$Q(1) = \{k : \text{ there is exactly one } \ell \text{ such that } r(x_\ell^*) \cap r(y_{j_k}) \neq \emptyset\},$$

$$Q(2) = \{k : \text{ there are at least two } \ell\text{'s such that } r(x_\ell^*) \cap r(y_{j_k}) \neq \emptyset\}.$$

For $k$'s in $Q(2)$ we have,

$$\begin{aligned}
& m_{j_0} \sum_\ell |\gamma_\ell| \Big| x_\ell^* \Big( \sum_{k \in Q(2)} (n_{j_0})_1^R(p_k) y_{j_k} \Big) \Big| \\
& \leq m_{j_0} \sum_{k \in Q(2)} (n_{j_0})_1^R(p_k) \sum_{\{\ell : r(x_\ell^*) \cap r(y_{j_k}) \neq \emptyset\}} |\gamma_\ell||x_\ell^*(y_{j_k})| \\
& \leq m_{j_0} \sum_{k \in Q(2)} (n_{j_0})_1^R(p_k) 10 \Big( \sum_{\{\ell : r(x_\ell^*) \cap r(y_{j_k}) \neq \emptyset\}} \gamma_\ell^2 \Big)^{\frac{1}{2}},
\end{aligned} \tag{13}$$

where for the second inequality we used that $y'_{j_k}s$ are smoothly normalized, and since, $(x_\ell^*)_\ell$ are $S_\xi$ admissible with $\xi < n_{j_0} < f_{2j_k}$ for all $k$, we applied Lemma 3.6. By the Cauchy-Schwartz inequality the estimate (13) continues as follows:

$$\begin{aligned}
& \leq 10 m_{j_0} \Big( \sum_{k \in Q(2)} ((n_{j_0})_1^R(p_k))^2 \Big)^{\frac{1}{2}} \Big( \sum_{k \in Q(2)} \sum_{\{\ell : r(x_\ell^*) \cap r(y_{j_k}) \neq \emptyset\}} \gamma_\ell^2 \Big)^{\frac{1}{2}} \\
& \leq 10 m_{j_0} \frac{2}{m_{j_0}^2} 2 \Big( \sum_{\{\ell : r(x_\ell^*) \cap r(u) \neq \emptyset\}} \gamma_\ell^2 \Big)^{\frac{1}{2}} \\
& \leq \frac{40}{m_{j_0}} \Big( \sum_{\{\ell \in B : r(x_\ell^*) \cap r(u) \neq \emptyset\}} \gamma_\ell^2 \Big)^{\frac{1}{2}} + \Big( \sum_{\{\ell \in E : \operatorname{supp} r(x_\ell^*) \cap r(u) \neq \emptyset\}} \gamma_\ell^2 \Big)^{\frac{1}{2}},
\end{aligned} \tag{14}$$

For the second inequality we used the fact that $(y_{j_k})_{k \in Q(2)}$ is $2S_\xi$ admissible for $\xi < n_{j_0}$, and that for every $\ell$ there are at most two values of $k \in Q(2)$ such that $r(x_\ell^*) \cap r(y_{j_k}) \neq \emptyset$.

For each $\ell$ let $s_\ell$ be the integer $s$ such that $m_{2j_s} \leq w(x_\ell^*) < m_{2j_{s+1}}$ and $r(x_\ell^*) \cap r(y_{j_k}) \neq \emptyset$ if such $s$ exists (obviously, no such $s$ exists if $\ell \in E$ i.e. is defined for certain values of $\ell \in B$). For $k$'s in $Q(1)$,

$$m_{j_0}\sum_{\ell}|\gamma_\ell|\Big|x_\ell^*\Big(\sum_{k\in Q(1)}(n_{j_0})_1^R(p_k)y_{j_k}\Big)\Big|$$
$$\le m_{j_0}\Big(\sum_{\ell}|\gamma_\ell|\Big|x_\ell^*\Big(\sum_{\{k\in Q(1):k\neq s_\ell\}}(n_{j_0})_1^R(p_k)y_{j_k}\Big)\Big|+\sum_{\ell}|\gamma_\ell|\Big|x_\ell^*\Big((n_{j_0})_1^R(p_{s_\ell})y_{j_{s_\ell}}\Big)\Big|\Big). \tag{15}$$

For the first term of the sum,

$$\begin{aligned}
&m_{j_0}\sum_{\ell}|\gamma_\ell|\Big|x_\ell^*\Big(\sum_{\{k\in Q(1):k\neq s_\ell\}}(n_{j_0})_1^R(p_k)y_{j_k}\Big)\Big|\\
&\le m_{j_0}\Big(\sum_{\ell\in B}|\gamma_\ell|\Big|x_\ell^*\Big(\sum_{\{k\in Q(1):k\neq s_\ell\}}(n_{j_0})_1^R(p_k)y_{j_k}\Big)\Big|+\sum_{\ell\in E}|\gamma_\ell|\Big|x_\ell^*\Big(\sum_{\{k\in Q(1):k\neq s_\ell\}}(n_{j_0})_1^R(p_k)y_{j_k}\Big)\Big|\Big)\\
&\le m_{j_0}\Big(\sum_{\ell\in B}|\gamma_\ell|\frac{5}{m_{j_0}^2}\Big(\sum_{\substack{k\in Q(1),k\neq s_\ell\\ r(x_\ell^*)\cap r(y_{j_k})\neq\emptyset}}((n_{j_0})_1^R(p_k))^2\Big)^{\frac12}+\sum_{\ell\in E}|\gamma_\ell|\frac{5}{m_{j_0}}\Big(\sum_{\substack{k\in Q(1),k\neq s_\ell\\ r(x_\ell^*)\cap r(y_{j_k})\neq\emptyset}}((n_{j_0})_1^R(p_k))^2\Big)^{\frac12}\Big)\\
&\le \frac{5}{m_{j_0}}\Big(\sum_{\ell\in B}\gamma_\ell^2\Big)^{\frac12}+5\Big(\sum_{\ell\in E}\gamma_\ell^2\Big)^{\frac12}.
\end{aligned}\tag{16}$$

For the second inequality of (16) we applied Lemma 3.9. Notice the $s_\ell$'s where picked to coincide with part (2) of Lemma 3.9. The final inequality of (16) followed from the Cauchy-Schwartz inequality.

For the second part of the right hand side of (15) notice that the only $\ell$'s that appear are the ones for which $s_\ell$ is defined. Also recall that if $s_\ell$ is defined then $w(x_\ell^*)>m_{j_0}$ hence $\ell\in B$. Thus the second part of the right hand side of (15) can be estimated as follows:

$$\begin{aligned}
m_{j_0}\sum_{\{\ell:\ s_\ell\ \text{is defined}\}}|\gamma_\ell|\Big|x_\ell^*\Big((n_{j_0})_1^R(p_{s_\ell})y_{j_{s_\ell}}\Big)\Big| &\le m_{j_0}\sum_{\{\ell:\ s_\ell\ \text{is defined}\}}|\gamma_\ell|(n_{j_0})_1^R(p_{s_\ell})\\
&\le m_{j_0}\Big(\sum_{\{\ell:\ s_\ell\ \text{is defined}\}}\gamma_\ell^2\Big)^{\frac12}\Big(\sum_{\{\ell:\ s_\ell\ \text{is defined}\}}((n_{j_0})_1^R(p_{s_\ell}))^2\Big)^{\frac12}\\
&\le \frac{2}{m_{j_0}}\Big(\sum_{\{\ell:\ s_\ell\ \text{is defined}\}}\gamma_\ell^2\Big)^{\frac12},
\end{aligned}\tag{17}$$

where for the second inequality we applied the Cauchy-Schwartz inequality and for the third inequality we used that $(x_\ell^*)_\ell$ is $S_\xi$ admissible for $\xi<n_{j_0}$ hence $\{(p_{s_\ell}):s_\ell \text{ is defined}\}\in 2S_\xi$. The result follows by combining the estimates (12), (13),(14),(15),(16), and (17). $\square$

**Lemma 3.12.** *Let $(y_j)_{j\in I}$, $j_0 \in 2\mathbb{N}$ and a normalized $(1/m_{j_0}^2, n_{j_0})$ squared average $g$ of $(y_j)_{j\in I}$ chosen as in Remark 3.10. Then for any $S_{n_i}$ admissible family $(x_\ell^*)_\ell \subset \mathcal{N}$, $i < j_0$ and $(\gamma_\ell)_\ell \in c_{00}$, we have*

$$\sum_\ell \gamma_\ell x_\ell^*(g) < \frac{123}{m_e}\Big(\sum_{\{\ell : w(x_\ell^*)\neq m_{j_0}, r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac{1}{2}} + 6\Big(\sum_{\{\ell : w(x_\ell^*)= m_{j_0}, r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac{1}{2}}, \tag{18}$$

*where $m_e = \min_\ell\{w(x_\ell^*), m_{j_0}\}$.*

*Proof.* Let $g = y/\|y\|$ and since $j_0$ is even note that $\|y\| \geq 1/m_{j_0}$ where $y = \sum_k (n_{j_0})_1^R(p_k) y_{j_k}$ for $p_k = \min\operatorname{supp} y_{j_k}$ for all $k \in \mathbb{N}$ and $R \in [(p_k)]$. Let $S = \{\ell : w(x_\ell^*) < m_{j_0}\}$, $E = \{\ell : w(x_\ell^*) = m_{j_0}\}$ and $B = \{\ell : w(x_\ell^*) > m_{j_0}\}$. Using Lemma 3.4 for $\ell \in S$ we write

$$x_\ell^* = \sum_{m\in L_\ell} \lambda_{\ell,m} x_{\ell,m}^*,$$

where $L_\ell = \cup_{i=1}^3 L_{\ell,i}$ and the following are satisfied:

$$\Big(\sum_{m\in L_{\ell,1}} \lambda_{\ell,m}^2\Big)^{\frac{1}{2}} \leq \frac{1}{m_{j_0}^2}, \Big(\sum_{m\in L_\ell} \lambda_{\ell,m}^2\Big)^{\frac{1}{2}} \leq \frac{1}{w(x_\ell^*)}, \tag{19}$$

$w(x_{\ell,m}^*) \geq m_{j_0}$ for $m \in L_{\ell,2}$, $x_{\ell,m}^* = \gamma_{\ell,m} e_{p_{\ell,m}}^*$ for $m \in L_{\ell,3}$, $|\gamma_{\ell,m}| \leq 1$ and $p_{\ell,m} \in \mathbb{N}$. Now we have

$$\begin{aligned}
\sum_\ell \gamma_\ell x_\ell^*(g) &\leq \Big|\sum_{\{\ell\in S : r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell \sum_{m\in L_\ell} \lambda_{\ell,m} x_{\ell,m}^*(g)\Big| + \Big|\sum_{\ell\notin S} \gamma_\ell x_\ell^*(g)\Big| \\
&\leq \Big|\sum_{\{\ell\in S : r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell \sum_{m\notin L_{\ell,2}} \lambda_{\ell,m} x_{\ell,m}^*(g)\Big| + \Big|\sum_{\{\ell\in S : r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell \sum_{m\in L_{\ell,2}} \lambda_{\ell,m} x_{\ell,m}^*(g)\Big| \\
&\quad + \frac{47}{m_{j_0}}\Big(\sum_{\{\ell\in B : r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac{1}{2}} + 6\Big(\sum_{\{\ell\in E : r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac{1}{2}}.
\end{aligned} \tag{20}$$

For the second inequality we applied Lemma 3.11. Now since $(x_{\ell,m}^*)_m$ is $S_{f_{j_0}}$ admissible and $(x_\ell^*)_\ell$ is $S_{n_i}$ admissible we can use the convolution property of Schreier families to conclude that $((x_{\ell,m}^*)_{\ell\in S, m\in L_\ell}$ is $S_{f_{j_0}+n_i}$ and hence $S_{2f_{j_0}}$ admissible. Thus for $\xi = 2f_{j_0} < n_{j_0}$ apply Lemma 3.11 to the second term of the sum to obtain,

$$
\begin{aligned}
&\Big|\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell\sum_{m\in L_{\ell,2}}\lambda_{\ell,m}x_{\ell,m}^*(g)\Big|\\
&\leq\frac{47}{m_{j_0}}\Bigg(\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell^2\sum_{\substack{m\in L_{\ell,2}\\ w(x_{\ell,m}^*)>m_{j_0}}}\lambda_{\ell,m}^2\Bigg)^{\frac12}+6\Bigg(\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell^2\sum_{\substack{m\in L_{\ell,2}\\ w(x_{\ell,m}^*)=m_{j_0}}}\lambda_{\ell,m}^2\Bigg)^{\frac12}\\
&\leq\frac{47}{m_{j_0}}\Bigg(\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell^2\Big(\frac{1}{w(x_\ell^*)}\Big)^2\Bigg)^{\frac12}+6\Bigg(\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell^2\Big(\frac{1}{w(x_\ell^*)}\Big)^2\Bigg)^{\frac12}\text{ (by (19))}\\
&\leq\frac{47}{m_{j_0}\min_\ell\{w(x_\ell^*)\}}\Bigg(\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell^2\Bigg)^{\frac12}+\frac{6}{\min_\ell\{w(x_\ell^*)\}}\Bigg(\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell^2\Bigg)^{\frac12}\\
&\leq\frac{53}{m_e}\Bigg(\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell^2\Bigg)^{\frac12}.
\end{aligned}\tag{21}
$$

Now separate the first term of (20) into two terms. Then recall $g=y/\|y\|$, $\|y\|\geq 1/m_{j_0}$, that $((x_{\ell,m}^*)_{\ell\in S,m\in L_\ell}$ is $S_{2f_{j_0}}$ admissible and apply Lemma 3.8 for "$\xi$"$=2f_{j_0}<n_{j_0}$,

$$
\text{“}\sum_m\gamma_m x_m^*\text{”}=\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell\sum_{m\in L_{\ell,1}}\lambda_{\ell,m}x_{\ell,m}^*
$$

and "$\sum_k\beta_k y_k$"$=y$ to conclude that,

$$
\begin{aligned}
&\Big|\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell\sum_{m\notin L_{\ell,2}}\lambda_{\ell,m}x_{\ell,m}^*(g)\Big|\\
&\qquad\leq\Big|\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell\sum_{m\in L_{\ell,1}}\lambda_{\ell,m}x_{\ell,m}^*(g)\Big|\\
&\qquad\quad+m_{j_0}\Big|\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell\sum_{m\in L_{\ell,3}}\lambda_{\ell,m}\gamma_{\ell,m}e_{p(\ell,m)}^*\sum_k(n_{j_0})_1^R(p_k)y_{j_k}\Big|\\
&\qquad\leq 22m_{j_0}\Bigg(\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell^2\sum_{m\in L_{\ell,1}}\lambda_{\ell,m}^2\Bigg)^{\frac12}\\
&\qquad\quad+m_{j_0}\Big|\sum_{\{\ell\in S:r(x_\ell^*)\cap r(g)\neq\emptyset\}}\gamma_\ell\sum_{m\in L_{\ell,3}}\lambda_{\ell,m}\gamma_{\ell,m}(n_{j_0})_1^R(p_{k(\ell,m)})\Big|,
\end{aligned}\tag{22}
$$

where for every $\ell\in S$ and $m\in L_{\ell,3}$, $k(\ell,m)$ is the unique integer $k$ (if any) such that $e_{p(\ell,m)}^*(y_{j_k})\neq 0$. If no such $k$ exists then the corresponding term is absent from the second part of the sum. Now (22) continues as follows by applying (19), the Cauchy-Schwartz inequality and the facts $|\gamma_{\ell,m}|\leq 1$ and $(k(\ell,m))_{\ell\in S,m\in L_{\ell,3}}\in S_{n_i+f_{j_0}}\subset S_{2f_{j_0}}$ where $2f_{j_0}<n_{j_0}$:

$$
\begin{aligned}
&\leq \frac{22m_{j_0}}{m_{j_0}^2}\Big(\sum_{\{\ell\in S: r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12}\\
&\quad + m_{j_0}\sum_{\{\ell\in S: r(x_\ell^*)\cap r(g)\neq\emptyset\}} |\gamma_\ell|\Big(\sum_{m\in L_{\ell,3}}\lambda_{\ell,m}^2\Big)^{\frac12}\Big(\sum_{m\in L_{\ell,3}}((n_{j_0})_1^R(p_{k(\ell,m)}))^2\Big)^{\frac12}\\
&\leq \frac{18}{m_{j_0}}\Big(\sum_{\{\ell\in S: r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12}\\
&\quad + \frac{m_{j_0}}{\min_\ell\{w(x_\ell^*)\}}\Big(\sum_{\{\ell\in S: r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12}\Big(\sum_{\{\ell\in S: r(x_\ell^*)\cap r(g)\neq\emptyset\}}\sum_{m\in L_{\ell,3}}((n_{j_0})_1^R(p_{k(\ell,m)}))^2\Big)^{\frac12} \qquad (23)\\
&\leq \frac{18}{m_{j_0}}\Big(\sum_{\{\ell\in S: r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12} + \frac{1}{m_{j_0}\min_\ell\{w(x_\ell^*)\}}\Big(\sum_{\{\ell\in S: r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12}\\
&\leq \frac{23}{m_{j_0}}\Big(\sum_{\{\ell\in S: r(x_\ell^*)\cap r(g)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12}.
\end{aligned}
$$

The result follows by combining (20),(21),(22) and (23). □

**Corollary 3.13.** *Let $(y_j)_{j\in I}$, $j_0\in 2\mathbb{N}$ and a normalized $(1/m_{j_0}^2, n_{j_0})$ squared average $g$ of $(y_j)_{j\in I}$ chosen as in Remark 3.10. Then for any $x^*\in\mathcal{N}$ such that $x^*(g)>1/2$ we have that $w(x^*)=m_{j_0}$.*

*Proof.* Let $x^*\in\mathcal{N}$ such that $x^*(g)>1/2$. Assume $w(x^*)\neq m_{j_0}$. Apply Lemma 3.12 for $i=0$, for (recall that $n_0=0$) the $S_{n_i}$ admissible family $(x_\ell^*)_\ell$ being the singleton $\{x^*\}$ and $\gamma_1=1$, to obtain that

$$x^*(g) < \frac{123}{m_e} \leq \frac{123}{m_1} < \frac12, \tag{24}$$

(where $m_e=\min(m_{j_0}, w(x^*))$), since $m_1>246$. □

## 4. $X$ is a Hereditarily Indecomposable Banach Space

We now show that $X$ is HI. We proceed by fixing $j\in\mathbb{N}$ and by defining vectors $(g_i)_{i=1}^p$ and $(z_i)_{i=1}^p$, functionals $(x_i^*)_{i=1}^p\in\mathcal{N}$, positive integers $(j_i)_{i=1}^p$ and $R=(t_i)\in\mathbb{N}$ which will be fixed throughout the section and shall be referred to in the results of the section. By using standard arguments we can assume that any two subspaces, in our case with trivial intersection, are spanned by normalized block bases of $(e_n)$. Let $(u_n)$ and $(v_n)$ be two such normalized block bases of $(e_n)$ and fix $j\in\mathbb{N}$. Set $P=(p_n)$ and $Q=(q_n)$ where $p_n=\min\operatorname{supp} u_n$ and $q_n=\min\operatorname{supp} v_n$ for all $n\in\mathbb{N}$. By passing to subsequences of $(p_n)$ and $(q_n)$ and relabeling, assume by (1) that if $R\in[P\cup Q]$ then for $\xi<n_{2j+1}$, $\sup\{(\sum_{k\in F}((n_{2j+1}))_1^R(k))^2)^{\frac12} : F\in S_\xi\} < 1/m_{2j+1}^2$. By Lemma 3.7 let $(y_j)_{j\in 2\mathbb{N}-1}$ (resp. $(y_j)_{j\in 2\mathbb{N}}$) be a block sequence of $(u_n)_n$ (resp. $(v_n)_n$) such that $y_j$ is a smoothly normalized

$(1/m_{2j}, f_{2j}+1)$ squared average of $(u_n)_n$ (resp. $(v_n)_n$). Apply Lemma 3.9 to $(y_j)_{j\in 2\mathbb{N}-1}$ and $(y_j)_{j\in 2\mathbb{N}}$ to obtain $I_1 \in [2\mathbb{N}-1]$ and $I_2 \in [2\mathbb{N}]$ such that $(y_j)_{j\in I_1}$ and $(y_j)_{j\in I_2}$ satisfy the statement of Lemma 3.9. For $j_1 \in \mathbb{N}$, $2j_1 > 2j+1$ let $g_1$ be a normalized $(1/m^2_{2j_1}, n_{2j_1})$ squared average of $(y_j)_{j\in I_1}$. Let $x_1^* \in \mathcal{N}$ with $x_1^*(g_1) > 1/2$ and $r(x_1^*) \subset r(g_1)$. By Corollary 3.13 we have that $w(x_1^*) = m_{2j_1}$. Let $m_{2j_2} = \sigma(x_1^*)$. Let $g_2$ be a normalized $(1/m^2_{2j_2}, n_{2j_2})$ squared average of $(y_j)_{j\in I_2}$ with $g_1 < g_2$. Let $x_2^* \in \mathcal{N}$ with $x_2^*(g_2) > 1/2$ and $r(x_2^*) \subset r(g_2)$. By Corollary 3.13 we have that $w(x_2^*) = m_{2j_2}$. Let $m_{2j_3} = \sigma(x_1^*, x_2^*)$. Let $g_3$ be a normalized $(1/m^2_{2j_3}, n_{2j_3})$ squared average of $(y_j)_{j\in I_1}$ with $g_1 < g_2 < g_3$. Let $x_3^* \in \mathcal{N}$ with $x_3^*(g_3) > 1/2$ and $r(x_3^*) \subset r(g_3)$. By Corollary 3.13 we have that $w(x_3^*) = m_{2j_3}$. Continue similarly to obtain $(g_i)_{i=1}^p$ and $(x_i^*)_{i=1}^p \subset \mathcal{N}$ such that:

(a) $g_i$ is a normalized $(1/m^2_{2j_i}, n_{2j_i})$ squared average of $(y_j)_{j\in I_1}$ for $i$ odd (resp. $(y_j)_{j\in I_2}$ for $i$ even).
(b) $w(x_i^*) = m_{2j_i}, r(x_i^*) \subset r(g_i)$ and $x_i^*(g_i) > 1/2$.
(c) $\sigma(x_1^*, \cdots, x_{i-1}^*) = w(x_i^*)$ for all $2 \le i \le p$.
(d) $\{g_i : i \le p\}$ is maximally $S_{n_{2j+1}}$- admissible.

Let $z_i = g_i/(x_i^*(g_i))$. Let $t_i = \min\operatorname{supp} z_i$ and $R = (t_i)_i$. The fact that $X$ is HI will follow from the next proposition.

**Proposition 4.1.** *For all $x^* \in \mathcal{N}$ there exist $J_1 < \cdots < J_s \subset \{1, \cdots, p\}$ such that,*

(1) $\{z_{minJ_m} : m \le s\} \in S_0 + 3S_{f_{2j+1}}$ *(i.e. it can be written as a union of four sets: one is a singleton and three belong to $S_{f_{2j+1}}$).*
(2) *There exists $(b_m)_{m=1}^s \subset \mathbb{R}$ such that*

$$\left| x^*\left(\sum_{i\in J_m}(-1)^i (n_{2j+1})_1^R(t_i) z_i\right)\right| \le (n_{2j+1})_1^R(t_{\min J_m}) b_m \text{ and } \left(\sum_m b_m^2\right)^{\frac{1}{2}} \le 6.$$

(3)

$$\left| x^*\left(\sum_{i\in\{1,\ldots,p\}\setminus\cup_{m=1}^s J_m}(-1)^i (n_{2j+1})_1^R(t_i) z_i\right)\right| \le \frac{505}{m^2_{2j+1}}.$$

Before presenting the proof of this proposition we show that it implies that $X$ is HI. First we find a lower estimate for $\|\sum_{i=1}^p (n_{2j+1})_1^R(t_i) z_i\|$.

$$\begin{aligned}\|\sum_i (n_{2j+1})_1^R(t_i) z_i\| &\ge \frac{1}{m_{2j+1}} \sum_k (n_{2j+1})_1^R(t_k) x_k^*\left(\sum_i (n_{2j+1})_1^R(t_i) z_i\right) \\ &= \frac{1}{m_{2j+1}} \sum_k ((n_{2j+1})_1^R(t_k))^2 = \frac{1}{m_{2j+1}}.\end{aligned}$$

Now we find an upper estimate for $\|\sum_{i=1}^p (-1)^i (n_{2j+1})_1^R(t_i) z_i\|$. Let $x^* \in \mathcal{N}$ and find $J_1 < J_2 < \cdots < J_s$ to satisfy Proposition 4.1.

$$
\begin{aligned}
x^*&\Big(\sum_i (-1)^i (n_{2j+1})_1^R(t_i) z_i\Big)\\
&\le \Big|x^*\Big(\sum_{i\in\cup_{m=1}^s J_m} (-1)^i (n_{2j+1})_1^R(t_i) z_i\Big)\Big| + \Big|x^*\Big(\sum_{i\notin\cup_{m=1}^s J_m} (-1)^i (n_{2j+1})_1^R(t_i) z_i\Big)\Big|\\
&\le \sum_{m=1}^s \Big|x^*\Big(\sum_{i\in J_m} (-1)^i (n_{2j+1})_1^R(t_i) z_i\Big)\Big| + \frac{505}{m_{2j+1}^2}\\
&\le \sum_{m=1}^s (n_{2j+1})_1^R(t_{\min J_m}) b_m + \frac{505}{m_{2j+1}^2}\\
&\le \Big(\sum_{m=1}^s ((n_{2j+1})_1^R(t_{\min J_m}))^2\Big)^{\frac12}\Big(\sum_{m=1}^s b_m^2\Big)^{\frac12} + \frac{505}{m_{2j+1}^2}\\
&\le \Big(4\Big(\frac{1}{m_{2j+1}^2}\Big)^2\Big)^{\frac12} 6 + \frac{505}{m_{2j+1}^2} = \frac{517}{m_{2j+1}^2}
\end{aligned}
$$

where the numbers "4" and "6" after the last inequality are justified by parts (1) and (2) of Proposition 4.1 respectively. Combining the two estimates we have that,

$$
\frac{517}{m_{2j+1}}\Big\|\sum_{i=1}^p (n_{2j+1})_1^R(t_i) z_i\Big\| \ge \frac{517}{m_{2j+1}^2} \ge \Big\|\sum_{i=1}^p (-1)^i (n_{2j+1})_1^R(t_i) z_i\Big\|,
$$

for any $j$ and thus $X$ is HI.

The remainder of the paper will be devoted to proving Proposition 4.1. The following three lemmas will be needed in the proof.

**Lemma 4.2.** *Let $x^* \in \mathcal{N}$ with $w(x^*) > m_{2j+1}$. Let $V \subset \{1, \ldots, p\}$ such that $w(x^*) \notin \{m_{2j_i} : i \in V\}$. Then*

$$
x^*\Big(\sum_{i\in V} \alpha_i z_i\Big) < \frac{496}{m_{2j+1}^2}\Big(\sum_{\{i\in V : r(x^*)\cap r(z_i)\neq\emptyset\}} \alpha_i^2\Big)^{\frac12}.
$$

*Proof.* Let $x^* = \frac{1}{w(x^*)}\sum_\ell \gamma_\ell y_\ell^*$ where $\sum_\ell \gamma_\ell^2 \le 1$ and $(y_\ell^*)_\ell$ is appropriately admissible. Define,

$$
Q(1) = \{i \in V : \text{there is exactly one } \ell \text{ such that } r(y_\ell^*) \cap r(z_i) \neq \emptyset\},
$$

$$
Q(2) = \{i \in V : \text{there is at least two } \ell\text{'s such that } r(y_\ell^*) \cap r(z_i) \neq \emptyset\}.
$$

For $i$'s in $Q(1)$,

$$
\begin{aligned}
x^*\Big(\sum_{i\in Q(1)} \alpha_i z_i\Big) &= \frac{1}{w(x^*)}\sum_\ell \gamma_\ell y_\ell^*\Big(\sum_{\{i\in Q(1): r(y_\ell^*)\cap r(z_i)\neq\emptyset\}} \alpha_i z_i\Big)\\
&\le \frac{1}{w(x^*)}\sum_\ell |\gamma_\ell| 2\sqrt{3}\Big(\sum_{\{i\in Q(1): r(y_\ell^*)\cap r(z_i)\neq\emptyset\}} \alpha_i^2\Big)^{\frac12}\\
&\le \frac{2\sqrt{3}}{w(x^*)}\Big(\sum_{\{i: r(x^*)\cap r(z_i)\neq\emptyset\}} \alpha_i^2\Big)^{\frac12} \le \frac{4}{m_{2j+1}^2}\Big(\sum_{\{i: r(x^*)\cap r(z_i)\neq\emptyset\}} \alpha_i^2\Big)^{\frac12},
\end{aligned}
\tag{25}
$$

where for the first inequality we used Proposition 3.2 and the fact that $x_i^*(g_i) > 1/2$, for the second inequality we applied the Cauchy-Schwartz inequality and for the last inequality we used the fact that if $2j+1 < i$ then $m_{2j+1}^2 \le m_i$. For the $Q(2)$ case, if $w(x^*) \neq m_{2j_i}$ for all $i \le p$ notice that

$$
\begin{aligned}
&x^*\Big(\sum_{\{i\in Q(2): r(x^*)\cap r(z_i)\neq\emptyset\}} \alpha_i z_i\Big)\\
&= \sum_{\{i\in Q(2): r(x^*)\cap r(z_i)\neq\emptyset\}} \alpha_i \bigg(\Big(\sum_{\{\ell: r(y_\ell^*)\cap r(z_i)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12} + \eta\bigg)\bigg|\frac{1}{w(x^*)}\sum_{\{\ell: r(y_\ell^*)\cap r(z_i)\neq\emptyset\}} \beta_{i,\ell} y_\ell^*(z_i)\bigg|
\end{aligned}
\tag{26}
$$

where $\beta_{i,\ell} = \gamma_\ell/((\sum_{\{m: r(y_m^*)\cap r(z_i)\neq\emptyset\}} \gamma_m^2)^{\frac12} + \eta_i)$ where $\eta_i$ is arbitrarily small and such that

$$
\Big(\sum_{\{m: r(y_m^*)\cap r(z_i)\neq\emptyset\}} \gamma_m^2\Big)^{\frac12} + \eta_i \in \mathbb{Q}.
$$

Now apply Lemma 3.12 for "$g$"$= g_i$, "$j_0$"$= 2j_i$, "$i$"$= 0$, "$\gamma_1$"$= 1$ and "$x_1^*$"$= \frac{1}{w(x^*)}\sum_{\{\ell: r(y_\ell^*)\cap r(z_i)\neq\emptyset\}} \beta_{i,\ell} y_\ell^* \in \mathcal{N}$ to continue (26) as follows:

$$
\begin{aligned}
&\le \sum_{\{i\in Q(2): r(y_\ell^*)\cap r(z_i)\neq\emptyset\}} |\alpha_i| \bigg(\Big(\sum_{\{\ell: r(x^*)\cap r(z_i)\neq\emptyset\}} \gamma_\ell^2\Big)^{\frac12} + \eta_i\bigg)\frac{2(123)}{w(x^*)}\\
&\le \frac{2(246)}{m_{2j+1}^2}\Big(\sum_{\{i\in V: r(x^*)\cap r(z_i)\neq\emptyset\}} \alpha_i^2\Big)^{\frac12},
\end{aligned}
\tag{27}
$$

where the constants $\eta_i$ were forgotten in the last inequality since they were arbitrarily small and the constant "123" that appears in the statement of Lemma 3.12 is multiplied by 2 since $z_i = g_i/x_i^*(g_i)$, $x_i^*(g_i) > 1/2$ and by another factor 2 since for each $\ell$ there are at most two values of $i \in Q(2)$ such that $r(y_\ell^*) \cap r(z_i) \neq \emptyset$. The result follows by combining (25), (26) and (27). □

**Lemma 4.3.** *Let $x^* \in \mathcal{N}$ with $w(x^*) = m_{2j+1}$. Thus $x^* = \frac{1}{m_{2j+1}}\sum_\ell \gamma_\ell y_\ell^*$ where $\sum_\ell \gamma_\ell^2 \le 1$, $y_\ell^* \in \mathcal{N}$, and $(y_\ell^*)_\ell$ has $S_{n_{2j+1}}$ dependent extension. Let $V \subset \{1 \le i \le p : m_{2j_i} \neq w(y_\ell^*)$ for all $\ell\}$ and $(\alpha_i)_i \in c_{00}$. Then*

$$x^*\Big(\sum_{i\in V}\alpha_i z_i\Big) < \frac{2}{m_{2j+1}^2}\Big(\sum_{\{i\in V: r(x^*)\cap r(z_i)\neq\emptyset\}}\alpha_i^2\Big)^{\frac{1}{2}}.$$

*Proof.* For $i\in I$ define $Q(1)$ and $Q(2)$ as in Lemma 4.2. For $i\in Q(1)$, use Lemma 4.2 for "$x^*$"$=y_\ell^*$ for each $\ell$ to obtain,

$$\begin{aligned}
x^*\Big(\sum_{i\in Q(1)}\alpha_i z_i\Big) &= \frac{1}{m_{2j+1}}\sum_\ell \gamma_\ell y_\ell^*\Big(\sum_{i\in Q(1)}\alpha_i z_i\Big)\\
&\le \frac{1}{m_{2j+1}^2}\sum_\ell|\gamma_\ell|\Big(\sum_{\{i\in V: r(y_\ell^*)\cap r(z_i)\neq\emptyset\}}\alpha_i^2\Big)^{\frac{1}{2}}\frac{496}{m_{2j+1}^2} \le \frac{1}{m_{2j+1}^2}\Big(\sum_{\{i\in V: r(x^*)\cap r(z_i)\neq\emptyset\}}\alpha_i^2\Big)^{\frac{1}{2}},
\end{aligned}\tag{28}$$

where for the last inequality we used the Cauchy Schwartz inequality and the fact that $m_{2j+1}\ge m_3\ge m_1^4\ge 496$. For $i\in Q(2)$,

$$\begin{aligned}
x^*\Big(\sum_{i\in Q(2)}\alpha_i z_i\Big) &= \frac{1}{m_{2j+1}}\sum_\ell\gamma_\ell y_\ell^*\Big(\sum_{i\in Q(2)}\alpha_i z_i\Big) \le \frac{1}{m_{2j+1}}\sum_{i\in Q(2)}|\alpha_i|\Big|\sum_\ell \gamma_\ell y_\ell^*(z_i)\Big|\\
&\le \frac{1}{m_{2j+1}}\sum_{i\in Q(2)}|\alpha_i|\Big(\sum_{\{\ell: r(y_\ell^*)\cap r(z_i)\neq\emptyset\}}\gamma_i^2\Big)^{\frac{1}{2}}\frac{2(123)}{\min_\ell\{w(y_\ell^*), m_{2j_i})\}}\\
&\le \frac{4(123)}{m_{2j+1}^3}\Big(\sum_{\{i: r(x^*)\cap r(z_i)\neq\emptyset\}}\alpha_i^2\Big)^{\frac{1}{2}} \le \frac{1}{m_{2j+1}^2}\Big(\sum_{\{i: r(x^*)\cap r(z_i)\neq\emptyset\}}\alpha_i^2\Big)^{\frac{1}{2}}.
\end{aligned}\tag{29}$$

Recall that $(y_\ell^*)_\ell$ is $S_{n_{2j+1}}$ admissible and $2j+1<2j_i$ for all $i$. Thus the second inequality follows from applying Lemma 3.12 for "$x_\ell^*$"$=y_\ell^*$, "$i$"$=2j+1$, "$j_0$"$=2j_i$, "$g$"$=g_i$ and observing that $z_i=g_i/x_i^*(g_i)$, $x^*(g_i)>1/2$ and $\{w(y_\ell^*):\ell\}\cap\{m_{2j_i}: i\in V\}=\emptyset$. The third inequality follows from applying the Cauchy-Schwartz inequality and observing that for every $\ell$ there are at most two values of $i\in Q(2)$ such that $r(y_\ell^*)\cap r(z_i)\neq\emptyset$ and $\min_{i,\ell}\{w(y_\ell^*), m_{2j_i}\}\ge m_{2j+1}^2$. The fourth inequality follows since $m_{2j+1}\ge m_3\ge m_1^4\ge 492$. The result follows by combining (28) and (29).

□

**Lemma 4.4.** *For $x^*\in\mathcal{N}$, $w(x^*)=m_{2j+1}$ there exist $J_1<J_2<J_3$ subsets of $\{1,\dots,p\}$ (some of which are possibly empty) such that:*

(1) *For $m\in\{1,2,3\}$,*

$$\Big|x^*\Big(\sum_{i\in J_m}(-1)^i(n_{2j+1})_1^R(t_i)z_i\Big)\Big|\le 2(n_{2j+1})_1^R(t_{minJ_m}).$$

(2) $x^*=\frac{1}{m_{2j+1}}\sum_{\ell=1}^n\gamma_\ell y_\ell^*$ *with* $\sum_\ell\gamma_\ell^2\le 1$, $y_\ell^*\in\mathcal{N}$, $(y_\ell^*)_\ell$ *has a* $S_{n_{2j+1}}$*-dependent extension and* $\{w(y_\ell^*):1\le\ell\le n\}\cap\{m_{2j_i}: i\in\{1,\dots,p\}\setminus\cup_{m=1}^3 J_m\}=\emptyset$.

(3) *Also,*

$$\Big|x^*\Big(\sum_{i\in\{1,\dots,p\}\setminus\cup_{m=1}^3J_m}(-1)^i(n_{2j+1})_1^R(t_i)z_i\Big)\Big|\le\frac{2}{m_{2j+1}^2}.$$

*Moreover, for any interval $Q \subset \{1,\dots,p\}$ there exist $J_1 < J_2 < J_3$ subsets of $Q$ (some of which are possibly empty) such that conditions (1), (2) and (3) are satisfied with the exception that in conditions (2) and (3) the set $\{1,\dots,p\} \setminus \cup_{m=1}^3 J_m$ is replaced by $Q \setminus \cup_{m=1}^3 J_m$.*

*Proof.* Suppose $x^* \in \mathcal{N}$ with $w(x^*) = m_{2j+1}$. Then $x^* = 1/m_{2j+1} \sum_{\ell=1}^n \gamma_\ell y_\ell^*$ with $\sum_\ell \gamma_\ell^2 \le 1$, $y_\ell^* \in \mathcal{N}$ and $(y_\ell^*)_{\ell=1}^n$ has a $S_{n_{2j+1}}$-dependent extension. Thus there exists $d, L \in \mathbb{N}$ and an $S_{n_{2j+1}}$ admissible family $(\tilde{y}_\ell^*)_{\ell=1}^{d+n-1}$ such that $\sigma(\tilde{y}_1^*,\dots,\tilde{y}_\ell^*) = w(\tilde{y}_{\ell+1}^*)$ for $1 \le \ell < d+n-1$ and $\tilde{y}_\ell^*|_{[L,\infty)} = y_{\ell-(d-1)}^*$ for $\ell = d,\dots,d+n-1$.

Recall the definition of $(x_k^*)$ from the beginning of this section. By injectivity of $\sigma$, the set $\{k \in \{1,\dots,p\} : w(x_k^*) \in \{w(\tilde{y}_\ell^*) : d \le i \le d+n-1\}\}$ is an interval of integers (possibly empty). Let $k_0$ be the largest integer $k$ such that $w(x_k^*) \in \{w(\tilde{y}_i^*) : d \le i \le d+n-1\}$ and $k_0 = 0$ if no such $k$ exists.

If $k_0 = 0$ (i.e. $w(x_k^*) \ne w(y_\ell^*)$ for all $1 \le \ell \le n, 1 \le k \le p$) then let $J_1 = J_2 = J_3 = \emptyset$ and conditions (1) and (2) are trivial. To verify condition (3) apply Lemma 4.3 for "$V$"$= \{k : 1 \le k \le p\}$ and "$\alpha_i$"$= (-1)^i (n_{2j+1})_1^R(t_i)$.

If $k_0 = 1$ (i.e. $w(x_1^*) = w(y_{i_0}^*)$ for some $i_0 \in \{1,\dots,n\}$ and $w(x_k^*) \notin \{w(y_\ell^*) : 1 \le \ell \le n\}$ for $1 < k \le p$) then set $J_1 = \{1\}$, $J_2 = J_3 = \emptyset$ and since $\|z_1\| = \|g_1/x_1^*(g_1)\| \le 2$ it is easy to check that conditions (1) and (2) hold. To check condition (3) apply Lemma 4.3 for "$V$"$= \{2,3,\dots,p\}$ and "$\alpha_i$"$= (-1)^i (n_{2j+1})_1^R(t_i)$.

If $k_0 > 1$ and $w(x_{k_0}^*) = w(\tilde{y}_1^*)$ then by the injectivity of $\sigma$, $w(x_k^*) \notin \{w(y_\ell^*) : 1 \le \ell \le n\}$ for $k \in \{1,\dots,p\} \setminus \{k_0\}$. Thus set $J_1 = \{k_0\}$, $J_2 = J_3 = \emptyset$ and easily verify conditions (1) and (2) as above. To check condition (3) apply Lemma 4.3 for "$V$"$= \{1,\dots,p\} \setminus \{k_0\}$.

Finally, if $k_0 > 1$ and $w(x_{k_0}^*) = w(\tilde{y}_{\ell_0}^*)$ for some $\ell_0 > 1$ by the injectivity of $\sigma$ it must be the case that $\ell_0 = k_0$, $w(\tilde{y}_i^*) = w(x_i^*)$ for all $i \le k_0$ and $\tilde{y}_i^* = x_i^*$ for all $i < k_0$. Let $J_1 = \{d\}$ if $d \le k_0$ (else $J_1 = J_2 = J_3 = \emptyset$), $J_2 = (d, k_0) \cap \mathbb{N}$ if $d < k_0$ (else $J_2 = J_3 = \emptyset$), $J_3 = \{k_0\}$ if $d < k_0$. By the choice of $J_1, J_2, J_3$ we have that $w(x_k^*) \notin \{w(y_\ell^*) : 1 \le \ell \le n\}$ for $k \in \{1,\dots,p\} \setminus \cup_{m=1}^3 J_m$ and if $J_2 \ne \emptyset$ then $x_i^* = \tilde{y}_i^* = y_{i-d+1}^*$ for $i \in J_2$. Apply Lemma 4.3 for "$V$"$= \{1,\dots,p\} \setminus \cup_{m=1}^3 J_m$ and "$\alpha_i$"$= (-1)^i (n_{2j+1})_1^R(t_i)$ to satisfy conditions (2) and (3).

To verify condition (1) for $J_1$ and $J_3$ (if they are non-empty), since they are singletons, simply observe that $|x^*(z_i)| \le \|z_i\| = \|g_i/x_i^*(g_i)\| \le 2$. To verify conditions (1) for $J_2$ (if it is non-empty),

$$
\begin{aligned}
\left| x^* \left( \sum_{i \in J_2} (-1)^i (n_{2j+1})_1^R(t_i) z_i \right) \right| &= \frac{1}{m_{2j+1}} \left| \sum_{i \in J_2} \gamma_{i-(d-1)} y_{i-(d-1)}^* \left( \sum_{i \in J_2} (-1)^i (n_{2j+1})_1^R(t_i) z_i \right) \right| \\
&= \frac{1}{m_{2j+1}} \left| \sum_{i \in J_2} \gamma_{i-(d-1)} x_i^* \left( \sum_{i \in J_2} (-1)^i (n_{2j+1})_1^R(t_i) z_i \right) \right| \\
&= \frac{1}{m_{2j+1}} \left| \sum_{i \in J_2} \gamma_{i-(d-1)} (-1)^i (n_{2j+1})_1^R(t_i) \right| \\
&\le \frac{\gamma_{\min J_2 - (d-1)}}{m_{2j+1}} (n_{2j+1})_1^R(t_{\min J_2}) \le (n_{2j+1})_1^R(t_{\min J_2}),
\end{aligned}
\tag{30}
$$

where the first two equalities follow from the fact that $x_i^* = y_{i-(d-1)}^*$ for $i \in J_2$; the third equality follows from the fact that $x_i^*(z_i) = x_i^*(g_i/x_i^*(g_i)) = 1$; the first inequality follows from the fact that $((n_{2j+1})_1^R(t_i))_i$ and $(\gamma_i)_i$ are both non-increasing sequences of non-negative numbers and $J_2$ is an interval.

The proof of the moreover part is identical to the above with the only exception that the sets $J_1, J_2, J_3$ chosen above are replaced by $Q \cap J_1, Q \cap J_2, Q \cap J_3$. Notice it was important in the proof of (30) that $J_2$ was an interval. This remains true if $J_2$ is replaced by $J_2 \cap Q$ since $Q$ was assumed to be an interval.

$\square$

*Proof of Proposition 4.1.* Suppose $w(x^*) > m_{2j+1}$, apply Lemma 4.2 for "$\alpha_i$"$= (-1)^i (n_{2j+1})_1^R(t_i)$. The conclusion of Propostion 4.1 is satisfied with $s = 1$ and $J_1 = \{q\}$ if $w(x^*) = m_{2j_q}$ or $J_1 = \emptyset$ if $w(x^*) \notin \{m_{2j_i} : 1 \leq i \leq p\}$.

If $w(x^*) = m_{2j+1}$ the proposition follows directly from Lemma 4.4 with $b_1 = b_2 = b_3 = 2$.

Assume $w(x^*) < m_{2j+1}$. Write $x^* = \sum_{\ell \in L} \gamma_\ell y_\ell^*$ where $(y_\ell^*)_\ell$, $(\lambda_\ell)_\ell$ and $L = L_1 \cup L_2 \cup L_3$ satisfy the conclusions of Lemma 3.4 for "$j$"$= 2j+1$. Let $L_2 = L_2' \cup L_2''$ where $L_2' = \{\ell \in L_2 : w(y_\ell^*) = m_{2j+1}\}$ and $L_2'' = \{\ell \in L_2 : w(y_\ell^*) > m_{2j+1}\}$. Define,

$$Q(1) = \{1 \leq i \leq p : \text{there is exactly one } \ \ell \in L \text{ with } r(y_\ell^*) \cap r(z_i) \neq \emptyset\},$$

$$Q(2) = \{1 \leq i \leq p : \text{there are at least two } \ell\text{'s in } L \text{ with } r(y_\ell^*) \cap r(z_i) \neq \emptyset\}.$$

For $i$'s in $Q(2)$,

$$\begin{aligned}
\Big|\Big(\sum_{\ell \in L} \lambda_\ell y_\ell^*\Big)\Big(\sum_{i \in Q(2)} (-1)^i (n_{2j+1})_1^R(t_i) z_i\Big)\Big| &\leq \sum_{i \in Q(2)} (n_{2j+1})_1^R(t_i) \Big|\sum_{\ell \in L} \lambda_\ell y_\ell^*(z_i)\Big| \\
&\leq 2 \sum_{i \in Q(2)} (n_{2j+1})_1^R(t_i) \Big|\sum_{\ell \in L} \lambda_\ell y_\ell^*(g_i)\Big| \\
&\leq 2(7) \sum_{i \in Q(2)} (n_{2j+1})_1^R(t_i) \Big(\sum_{\{\ell \in L : r(y_\ell^*) \cap r(z_i) \neq \emptyset\}} \lambda_\ell^2\Big)^{\frac{1}{2}} \\
&\leq 14 \Big(\sum_{i \in Q(2)} ((n_{2j+1})_1^R(t_i))^2\Big)^{\frac{1}{2}} \Big(\sum_{i \in Q(2)} \sum_{\{\ell \in L : r(y_\ell^*) \cap r(z_i) \neq \emptyset\}} \lambda_\ell^2\Big)^{\frac{1}{2}} \\
&\leq 14 \frac{2}{m_{2j+1}^2} \frac{2}{w(x^*)} \leq \frac{1}{m_{2j+1}^2},
\end{aligned} \tag{31}$$

where for the second inequality we used the definition of $z_i$. For the third inequality we applied Lemma 3.12 for "$j_0$"$= 2j_i$ and "$g$"$= g_i$ for each $i$ using the fact that $(y_\ell^*)_\ell$ is $S_{f_{2j+1}}$ admissible, $f_{2j+1} < n_{2j+1}$ and $2j+1 < 2j_i$ and noticing that the right hand side of (18) is at most equal to $7(\sum_{\{\ell : r(x_\ell^*) \cap r(g) \neq \emptyset\}} \gamma_\ell^2)^{\frac{1}{2}}$ since $103/m_e \leq 1$. For the fourth inequality we used the Cauchy-Schwartz inequality. For the fifth inequality we used that $(t_i)_{i \in Q(2)}$ is $2S_{f_{2j+1}}$ is

admissible, condition (2) of Lemma 3.4, and the fact that for every $\ell \in L$ there are at most two values of $i \in Q(2)$ such that $r(y_\ell^*) \cap r(z_i) \neq \emptyset$. Now for $\ell$'s in $L_1$ and $i$'s in $Q(1)$,

$$\begin{aligned}
&\Big|\Big(\sum_{\ell\in L_1}\lambda_\ell y_\ell^*\Big)\Big(\sum_{\substack{i\in Q(1)\\ r(y_\ell^*)\cap r(z_i)\neq\emptyset}}(-1)^i(n_{2j+1})_1^R(t_i)z_i\Big)\Big| \\
&\qquad\le 2\Big|\Big(\sum_{\ell\in L_1}\lambda_\ell y_\ell^*\Big)\Big(\sum_{\substack{i\in Q(1)\\ r(y_\ell^*)\cap r(z_i)\neq\emptyset}}(-1)^i(n_{2j+1})_1^R(t_i)g_i\Big)\Big| \\
&\qquad\le 2\sqrt{3}\sum_{\ell\in L_1}|\lambda_\ell|\Big(\sum_{\substack{i\in Q(1)\\ r(y_\ell^*)\cap r(z_i)\neq\emptyset}}((n_{2j+1})_1^R(t_i))^2\Big)^{\frac{1}{2}} \le \frac{4}{m_{2j+1}^2},
\end{aligned} \tag{32}$$

where Proposition 3.2 was used in the second inequality and for the last inequality we used the Cauchy-Schwartz inequality and condition (2) of Lemma 3.4. For $\ell \in L_3$ there is at most one value of $i$ (call it $i(\ell)$) such that $r(y_\ell^*) \cap r(z_i) \neq \emptyset$. Thus

$$\begin{aligned}
\Big|\Big(\sum_{\ell\in L_3}\lambda_\ell y_\ell^*\Big)\Big(\sum_{\{i\in Q(1):r(y_\ell^*)\cap r(z_i)\neq\emptyset\}}(-1)^i(n_{2j+1})_1^R(t_i)z_i\Big)\Big| &\le \sum_{\ell\in L_3}|\lambda_\ell|(n_{2j+1})_1^R(t_{i(\ell)})2| \\
&\le \frac{2}{m_{2j+1}^2},
\end{aligned} \tag{33}$$

where for the first inequality we used that $|y_\ell^*(z_i)| \le \|z_i\| \le 2$ and for the second inequality we applied the Cauchy-Schwartz inequality and used that $(t_{i(\ell)})_{\ell\in L}$ is $S_{f_{2j+1}}$ admissible and $f_{2j+1} < n_{2j+1}$. For $\ell'' \in L_2''$, set $J_{\ell''} = \emptyset$ if $w(y_{\ell''}^*) \notin \{m_{2j_i} : 1 \le i \le p\}$ and $J_{\ell''} = \{q\}$ if $w(y_{\ell''}^*) = m_{2j_q}$ for some $q \in \{1,\dots,p\}$ and $r(y_{\ell''}^*) \cap r(z_q) \neq \emptyset$. Notice that for $\ell'' \in L_2''$ if $J_{\ell''} = \{q\} \neq \emptyset$ then we have,

$$\begin{aligned}
\Big|x^*\Big(\sum_{i\in J_{\ell''}}(-1)^i(n_{2j+1})_1^R(t_i)z_i\Big)\Big| &\le (n_{2j+1})_1^R(t_q)|x^*(z_q)| \\
&= (n_{2j+1})_1^R(t_q)|\lambda_{\ell''}y_{\ell''}^*(z_q)| \\
&\le (n_{2j+1})_1^R(t_q)|\lambda_{\ell''}|2 \quad (\text{since } \|z_q\| \le 2).
\end{aligned} \tag{34}$$

For $\ell' \in L_2'$ apply the moreover part of Lemma 4.4 for "$x^*$"$= y_{\ell'}^*$ and "$Q$"$= Q_{\ell'} = \{i \in Q(1) : r(z_i) \cap r(y_{\ell'}^*) \neq \emptyset\}$ to obtain intervals $J_{\ell',1} < J_{\ell',2} < J_{\ell',3} \subset Q_{\ell'}$ (possibly empty) such that

$$\Big|y_{\ell'}^*\Big(\sum_{i\in J_{\ell',m}}(-1)^i(n_{2j+1})_1^R(t_i)z_i\Big)\Big| \le 2(n_{2j+1})_1^R(t_{\min J_{\ell',m}}), \quad m \in \{1,2,3\} \tag{35}$$

and

$$y^*_{\ell'} = \frac{1}{m_{2j+1}} \sum_k \gamma_{\ell',k} y^*_{\ell',k} \text{ with } \sum_k \gamma^2_{\ell',k} \leq 1, y^*_{\ell',k} \in \mathcal{N}, (y^*_{\ell',k})_k \text{ is } S_{n_{2j+1}} \text{ admissible and} \tag{36}$$
$$\{w(y^*_{\ell',k}) : k\} \cap \{m_{2j_i} : i \in Q_{\ell'} \setminus \cup_{m=1}^3 J_{\ell',m}\} = \emptyset.$$

Thus for $\ell' \in L'_2$ and $m \in \{1,2,3\}$ apply the fact $J_{\ell',m} \subset Q_{\ell'} := \{i \in Q(1) : r(z_i) \cap r(y^*_{\ell'}) \neq \emptyset\}$ and (35) to obtain

$$\left| x^* \Big( \sum_{i \in J_{\ell',m}} (-1)^i (n_{2j+1})_1^R (t_i) z_i \Big) \right| \leq |\lambda_{\ell'}| \left| y^*_{\ell'} \Big( \sum_{i \in J_{\ell,m}} (-1)^i (n_{2j+1})_1^R (t_i) z_i \Big) \right| \tag{37}$$
$$\leq (n_{2j+1})_1^R (t_{min J_{\ell',m}}) |\lambda_{\ell'}| 2.$$

Thus for $\ell'' \in L''_2$ if $J_{\ell''} \neq \emptyset$, set $b_{\ell''} = 2\lambda_{\ell''}$ and for $\ell' \in L'_2$ and $m \in \{1,2,3\}$ if $J_{\ell',m} \neq \emptyset$ set $b_{\ell',m} = 2\lambda_{\ell'}$. Hence,

$$\sum_{\{\ell'' \in L''_2 : J_{\ell''} \neq \emptyset\}} b^2_{\ell''} + \sum_{\{\ell' \in L'_2, m \in \{1,2,3\} : J_{\ell',m} \neq \emptyset\}} b^2_{\ell',m} \leq \sum_{\{\ell'' \in L''_2 : J_{\ell''} \neq \emptyset\}} 4\lambda^2_{\ell''} + \sum_{\{\ell' \in L'_2, m \in \{1,2,3\} : J_{\ell',m} \neq \emptyset\}} 4\lambda^2_{\ell'}$$
$$\leq 16 \sum_{\ell \in L} \gamma^2_\ell \leq \frac{16}{w(x^*)^2} \leq 1, \tag{38}$$

where the third inequality was obtained from condition (2) of Lemma 3.4. Index the $b_{\ell''}$ (for $\ell'' \in L''_2$ with $J_{\ell''} \neq \emptyset$) and $b_{\ell',m}$ (for $\ell' \in L'_2$ and $m \in \{1,2,3\}$ with $J_{\ell',m} \neq \emptyset$) as $(b_m)_{m=1}^s$ the non-empty $J_{\ell''}$ ($\ell'' \in L''_2$) and $J_{\ell',m}$ ($\ell' \in L'_2, m \in \{1,2,3\}$) as $(J_m)_{m=1}^s$ and we can see that (34), (37) and (38) imply that condition (2) of Proposition 4.1 is satisfied. Condition (1) of Proposition 4.1 follows from the facts that $(y^*_\ell)_{\ell \in L}$ is $S_{f_{2j+1}}$ admissible, for every $\ell'' \in L''_2$ for which $J_{\ell''} = \{q\} \neq \emptyset$ we have $r(z_q) \cap r(y^*_{\ell'}) \neq \emptyset$ and for every $\ell' \in L'_2$ and $m \in \{1,2,3\}$ such that $J_{\ell',m} \neq \emptyset$ we have $J_{\ell',m} \subset \{i \in Q(1) : r(z_i) \cap r(y^*_{\ell'}) \neq \emptyset\}$ (for $m = 1,2,3$). The next Lemma 4.5 implies that

$$\left| \Big( \sum_{\ell \in L_2} \lambda_\ell y^*_\ell \Big) \Big( \sum_{i \notin \cup_{m=1}^s J_m} (-1)^i (n_{2j+1})_1^R (t_i)(z_i) \Big) \right| \leq \frac{498}{m^2_{2j+1}}. \tag{39}$$

Indeed apply Lemma 4.5 for "$E$"$= L'_2$, $B = L''_2$, "$V$"$= Q(1) \setminus \cup_{m=1}^s J_m$, "$\alpha_i$"$= (-1)^i (n_{2j+1})_1^R (t_i)$. Note that condition (1) of Lemma 4.5 is satisfied by (36). Condition (2) of Lemma 4.5 is satisfied by the definitions of $L''_2$ and $J_{\ell''}$ for $\ell'' \in L''_2$. Condition (3) of Lemma 4.5 is satisfied since $V \subset Q(1)$. Thus equations (31), (32),(33) and (39) imply condition (3) and finish the proof of Proposition 4.1.

*Q.E.D.*

**Lemma 4.5.** *Let $E$ and $B$ be two finite index sets, $V \subset \{1, \dots, p\}$, $(y^*_\ell)_{\ell \in E \cup B} \subset \mathcal{N}$, $(\lambda_\ell)_{\ell \in E \cup B}$ be scalars with $\sum_{\ell \in E \cup B} \lambda^2_\ell \leq 1$ and $(\alpha_i)_{i \in V}$ be scalars with $\sum_{i \in V} \alpha^2_i \leq 1$ such that,*

(1) *For every* $\ell \in E$, $y_\ell^*$ *can be written as*

$$y_\ell^* = \frac{1}{m_{2j+1}} \sum_{\ell=1}^{r_\ell} \gamma_{\ell,s} y_{\ell,s}^*$$

*with* $y_{\ell,s}^* \in \mathcal{N}$, $(y_{\ell,s}^*)_{\ell=1}^{r_\ell}$ *is* $S_{n_{2j+1}}$*-admissible and* $\{w(y_{\ell,s}^*) : 1 \leq s \leq r_\ell\} \cap \{m_{2j_i} : i \in V\} = \emptyset$.

(2) *For every* $\ell \in B$, $w(y_\ell^*) > m_{2j+1}$ *and* $w(y_\ell^*) \notin \{m_{2j_i} : i \in V\}$.

(3) *For every* $k \in V$ *there is a unique* $\ell \in E \cup B$ *such that* $r(y_\ell^*) \cap r(z_i) \neq \emptyset$.

*Then,*

$$\left| \Big( \sum_{\ell \in E \cup B} \lambda_\ell y_\ell^* \Big) \Big( \sum_{i \in V} \alpha_i z_i \Big) \right| \leq \frac{498}{m_{2j+1}^2}.$$

*Proof.* Let $E$, $B$, $(\lambda_\ell)_{\ell \in E \cup B}$, $(y_\ell^*)_{\ell \in E \cup B}$ and $V$ be given as in the hypothesis. Using the triangle inequality we separate into the cases $\ell \in B$ and $\ell \in E$. For $\ell$'s in $B$,

$$\begin{aligned} \left| \Big( \sum_{\ell \in B} \lambda_\ell y_\ell^* \Big) \Big( \sum_{\{i \in V : r(y_\ell^*) \cap r(z_i) \neq \emptyset\}} \alpha_i z_i \Big) \right| &\leq \sum_{\ell \in B} |\lambda_\ell| \frac{496}{m_{2j+1}^2} \Big( \sum_{\{i \in V : r(y_\ell^*) \cap r(z_i) \neq \emptyset\}} \alpha_i^2 \Big)^{\frac{1}{2}} \\ &\leq \frac{496}{m_{2j+1}^2}, \end{aligned}$$

where for the first inequality we used Lemma 4.2 for "$x^*$"$= y_\ell^*$, noting that its hypothesis is satisfied by our assumption (2) and for the second inequality we used the Cauchy-Schwartz inequality. For $\ell$'s in $E$,

$$\begin{aligned} \left| \Big( \sum_{\ell \in E} \lambda_\ell y_\ell^* \Big) \Big( \sum_{\{i \in V : r(y_\ell^*) \cap r(z_i) \neq \emptyset\}} \alpha_i z_i \Big) \right| &\leq \sum_{\ell \in E} |\lambda_\ell| \frac{2}{m_{2j+1}^2} \Big( \sum_{\{i \in V : r(y_\ell^*) \cap r(z_i) \neq \emptyset\}} \alpha_i^2 \Big)^{\frac{1}{2}} \\ &\leq \frac{2}{m_{2j+1}^2}. \end{aligned}$$

where the first inequality follows by applying Lemma 4.3 for each $i$ with "$x^*$"$= y_i^*$, noting that its hypothesis is satisfied by our assumption (1), and for the second inequality we used the Cauchy-Schwartz inequality.

□

## References


[1] D. Alspach and S.A. Argyros, *Complexity of weakly null sequences*, Dissertationes Math. 321 (1992).

[2] S.A. Argyros and I. Deliyanni, *Examples Of asymptotic $\ell_1$ Banach spaces*, Trans. Amer. Math. Soc. 349 (1997), 973-995.

[3] S.A. Argyros and V. Felouzis, *Interpolating Hereditarily Indecomposable Banach spaces*, J. Amer. Math. Soc. 13:2, 243-294.

[4] S.A. Argyros and A. Manoussakis, *An Indecomposable and Unconditionally saturated Banach space*, Studia Math. **159** (1) (2003), 1-32.

[5] S.A. Argyros and A. Manoussakis, *A sequentially unconditional Banach space with few operators*, Proc. LMS (to appear).

[6] S.A. Argyros and A. Tolias, *Indecomposability and unconditionality in duality*, Geom. Funt. Anal. **14** (2004) no 2, 247-282.

[7] N. Aronszajn and K.T. Smith, *Invariant subspaces of completely continuous operators* Ann. of Math. (2), 60 (1954), 345-350.

[8] N. Dew, *Asymptotic Structure of Banach Spaces*, Ph.D dissertation, Univerisity of Oxford, St. John's College (2002).

[9] I. Gasparis, *A continuum of totally imcomparable Hereditarily Hndecomposable Banach spaces*, Studia Math. **151** (3) (2002), 277-298.

[10] I. Gasparis and D.H. Lueng, *On the complemented subspaces of the Schreier spaces*, Studia Math. 131 (2000), 273-300.

[11] W.T. Gowers, *A remark about the scalar-plus-compact problem*, Convex geometric analysis (Berkeley, CA, 1996), 111-115, Math. Sci. Res. Inst. Publ. 34, Cambridge Univ. Press, Cambridge, 1999.

[12] W.T. Gowers, *A Hereditarily Indecomposable space with an asymptotic unconditional basis*, Geometric aspects of functional analysis (Israel, 1992-1994), 112-120, Oper. Theory Adv. Appl.,77, Birkhäuser, Basel 1995.

[13] W.T. Gowers, B. Maurey *The unconditional basic sequence problem* Journal of AMS **6** (1993), no. 4, 851-874.

[14] W.T. Gowers, B. Maurey *Banach spaces with small spaces of operators* Math. Ann. **307** (1997), no. 4, 543-568.

[15] V. Ferenczi, *Operators on subspaces of Hereditarily Indecomposable Banach spaces*, Bull. London Math. Soc. **29**, (1997) 338-344.

[16] V. Ferenczi, *A uniformly convex Hereditarily Indecomposable Banach space*, Israel J. Math. **102**, (1997) 199-225.

[17] B. Maurey, *A remark about distortion* Geometric aspects of functional analysis (Israel, 1992-1994), 131-142, Oper. Theory Adv. Appl.,77, Birkhäuser, Basel 1995.

[18] G. Pisier, *Weak Hilbert spaces*, Proc. London. Math. Soc. (3) 56 (1988), no. 3, 547-579.

[19] G. Pisier, *The volume of convex bodies and Banach space geometry*, Cambridge Tracts in Mathematics, 94. Cambridge University Press, Cambridge, 1989.

Department of Mathematics, University of South Carolina, Columbia, SC 29208,

giorgis@math.sc.edu, beanland@math.sc.edu